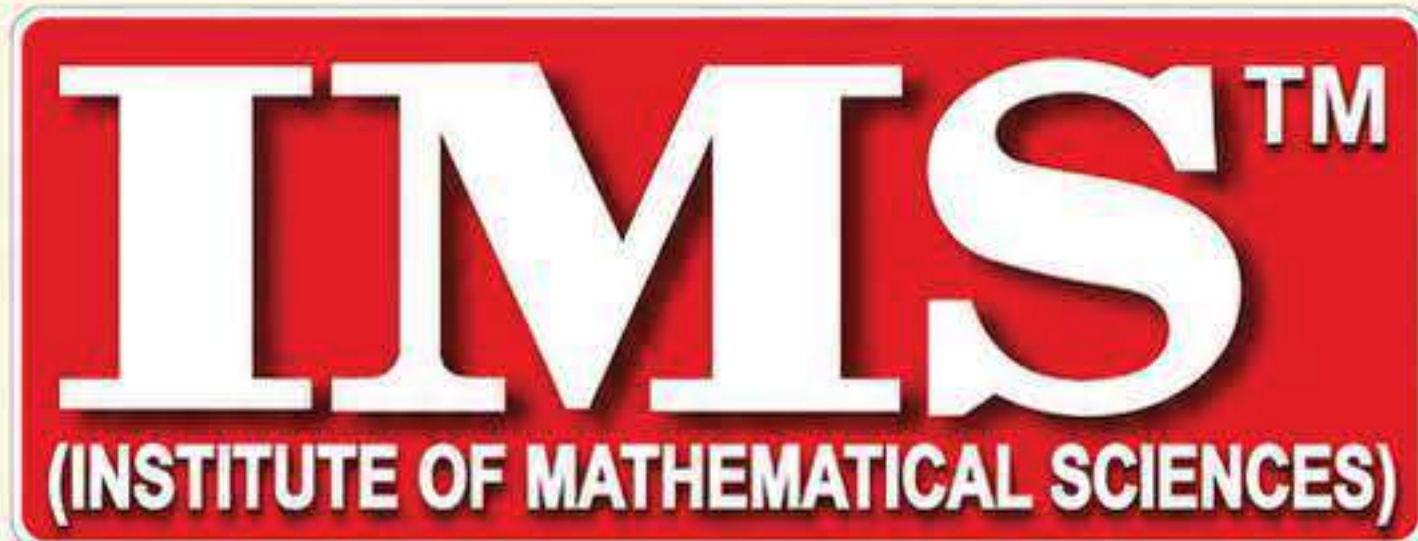




UPSC-CSE 2026

Mains

MATHEMATICS

Optional Paper-I

Solutions

1(a) Find all the values of 'a' for which the system of homo. equations

$$ax + y + z = 0$$

$$x + y - z = 0$$

$$x + y - az = 0$$

has non-trivial solutions, and hence determine all the solutions for which each value of 'a'.

Sol

we have

$$\begin{bmatrix} a & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & -a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow AX = 0 \quad \text{--- (1) say}$$

where $A = \begin{bmatrix} a & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & -a \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

If $|A| = 0$ the given system of equations have non-trivial solutions for x, y, z .

we have $\begin{vmatrix} a & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & -a \end{vmatrix} = 0$

1. (b) $\rightarrow$ If A and B are similar matrices then show that A and B have the same rank, trace, characteristic polynomial and eigen values.

sol Given that A and B are similar matrices

$\therefore$ By definition:

$\exists$ non-singular matrix C

i.e. $|C| \neq 0$ s.t.

$$B = C^{-1}AC \quad \text{--- (1)}$$

Rank:

Multiplying by invertible matrices does not change rank.

$$\therefore r(B) = r(C^{-1}AC) = r(A) \\ \Rightarrow r(A) = r(B)$$

characteristic polynomial:

$$\text{We have } B - \lambda I = C^{-1}AC - \lambda I$$

$$\Rightarrow |B - \lambda I| = |C^{-1}AC - \lambda I|$$

$$= |C^{-1}AC - C^{-1}(\lambda I)C|$$

$$= |C^{-1}(A - \lambda I)C|$$

$$= |C^{-1}| |A - \lambda I| |C|$$

$$\Rightarrow a(-a+1) - 1(-a+1) + 1(1-1) = 0$$

$$\Rightarrow (-a+1)(a-1) = 0$$

$$\Rightarrow (a-1)^2 = 0 \Rightarrow a = 1, 1.$$

$\therefore$ If $a = 1$ the given system of equations have non-zero solutions to x, y, z

$\therefore a = 1$, the given system of equations reduces to

$$\begin{cases} x+y+z=0 \\ x+y-z=0 \\ x+y-z=0 \end{cases} \Rightarrow y = -x \text{ and } z = 0$$

$\therefore$ All the non-trivial solutions are given by $(x, y, z) = (x, -x, 0) = x(1, -1, 0)$

If $a \neq 1$ then we get $(x, y, z) = (0, 0, 0)$ which is known as trivial solution.

$$\Rightarrow |B - \lambda I| = |A - \lambda I| |\bar{C}^T| |C|$$

$$\Rightarrow |B - \lambda I| = |A - \lambda I| |\bar{C}^T C|$$

$$\Rightarrow |B - \lambda I| = |A - \lambda I| \quad (\because |\bar{C}^T C| = 1)$$

$\therefore A$ and B have
same characteristic polynomial.

$\therefore$ They have same
characteristic roots (eigen value)

Trace: $\because B = \bar{C}^T A C$

$$\Rightarrow \text{Trace}(B) = \text{Trace}(\bar{C}^T A C)$$

$$\Rightarrow \text{Trace}(B) = \text{Trace}(A C \bar{C}^T)$$

(by using
 cyclic property
 of trace)

$$\Rightarrow \text{Trace}(B) = \text{Trace}(A)$$

=====

1(c) If $u(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$, then find the value of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$.

Solⁿ: Let $r^2 = x^2 + y^2 + z^2$ — (1)

then $u = r^{-1}$ — (2)

$$\therefore 2r \frac{\partial r}{\partial x} = 2x$$

$$\Rightarrow \boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}$$

$$\text{Also } \frac{\partial u}{\partial r} = -\frac{1}{r^2}$$

$$\therefore \text{ we have } \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x}$$

$$= -\frac{1}{r^2} \left[\frac{x}{r} \right]$$

$$= -\frac{x}{r^3}$$

$$\therefore \boxed{\frac{\partial u}{\partial x} = -\frac{x}{r^3}}$$

we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = -\frac{\partial}{\partial x} \left(\frac{x}{r^3} \right)$$

$$= - \left[r^3(1) - x(3)r^2 \frac{\partial r}{\partial x} \right]$$

$$= -\frac{1}{r^3} + \frac{3x}{r^4} \left(\frac{x}{r} \right) = -\frac{1}{r^3} + \frac{3x^2}{r^5}$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{1}{x^3} + \frac{3x^2}{x^5}$$

$$\text{Similarly } \frac{\partial^2 u}{\partial y^2} = -\frac{1}{y^3} + \frac{3y^2}{y^5}$$

$$\text{and } \frac{\partial^2 u}{\partial z^2} = -\frac{1}{z^3} + \frac{3z^2}{z^5}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} =$$

$$\left(-\frac{1}{x^3} + \frac{3x^2}{x^5}\right) + \left(-\frac{1}{y^3} + \frac{3y^2}{y^5}\right)$$

$$+ \left(-\frac{1}{z^3} + \frac{3z^2}{z^5}\right)$$

$$= -\frac{3}{x^3} + \frac{3}{x^5} (x^2 + y^2 + z^2)$$

$$= -\frac{3}{x^3} + \frac{3x^2}{x^5} = 0$$

1. (d) →

Using Maclaurin's theorem, obtain the first four terms of the expansion of $e^{x \cos x}$ in ascending powers of 'x'. Hence or otherwise find the limit of $\frac{e^x - e^{x \cos x}}{x - \sin x}$ as x tends to zero.

sol Let $f(x) = e^{x \cos x}$ — (1)

We know that Maclaurin's theorem:

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(iv)}(0) + \dots$$
 — (2)

We have

$$f(x) = e^{x \cos x} \Rightarrow f(0) = 1$$

$$f'(x) = e^{x \cos x} (x \sin x + \cos x)$$

$$\Rightarrow f'(0) = 1$$

$$\text{slly } f''(0) = 1 \text{ and } f'''(0) = -2$$

$$\therefore e^{x \cos x} = 1 + x + \frac{x^2}{2} - \frac{x^3}{3} + \dots$$

We know that

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

$$\therefore x - \sin x = \frac{x^3}{6} - \frac{x^5}{120} + \dots$$

$$e^x - e^{x \cos x} = \frac{x^3}{6} - \left(\frac{x^3}{6} \right) + o(x^4)$$

$$\text{Also } x - \sin x = \frac{x^3}{6} + o(x^5)$$

$$\therefore \lim_{x \rightarrow 0} \left(\frac{e^x - e^{x \cos x}}{x - \sin x} \right) = \lim_{x \rightarrow 0} \frac{\frac{x^3}{6}}{\frac{x^3}{6}} = 1$$

1(e) $\rightarrow$ Obtain the equation of the sphere having its centre on the line $5y+2z=0=2x-3y$ and passing through the points $(0, -2, -4)$ and $(2, -1, -1)$.

Solⁿ: Let the equation of the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad \text{--- (1)}$$

Since its centre lies on the line

$$5y + 2z = 0 = 2x - 3y$$

$$\therefore 5(-v) + 2(-w) = 0 = 2(-u) - 3(-v)$$

$$\text{i.e., } 5v + 2w = 0 \quad \text{--- (2)}$$

$$\& \quad 2u - 3v = 0 \quad \text{--- (3)}$$

Since the sphere passes through the points $(0, -2, -4)$ and $(2, -1, -1)$.

$$\therefore 0 + 4 + 16 + 0 - 4v - 8w + d = 0$$

$$\Rightarrow -4v - 8w + d + 16 = 0 \quad \text{--- (4)}$$

$$\& 4 + 1 + 1 + 4u - 2v - 2w + d = 0$$

$$\Rightarrow 4u - 2v - 2w + d + 6 = 0 \quad \text{--- (5)}$$

Solving the eqns (2), (3), (4) & (5)

we get $u = -3, v = -2, w = 5, d = 12$

$\therefore$ (1) $\equiv$

$$x^2 + y^2 + z^2 - 6x - 4y + 10z + 12 = 0$$

which is the required equation.

2(a). Let $T: P_2[x] \rightarrow P_2[x]$ be a linear transformation defined by

$$T(a_0 + a_1x + a_2x^2) = -(a_0 + 2a_1 + a_2) + (2a_0 + 3a_1)x - 2(a_0 + a_1)x^2.$$

Where $P_2[x]$ denotes the set of all polynomials in x of degree ≤ 2 and $a_0, a_1, a_2 \in \mathbb{R}$.

Find the eigenvalues and corresponding eigenvectors of T using the matrix representation of T w.r.t standard basis of $P_2[x]$.

Sol Let $P_2[x] = \left\{ p(x)/p(x) = a_0 + a_1x + a_2x^2 \mid a_0, a_1, a_2 \in \mathbb{R} \right\}$

Given that $T: P_2[x] \rightarrow P_2[x]$

s.t $T(a_0 + a_1x + a_2x^2) =$

$$-(a_0 + 2a_1 + a_2) + (2a_0 + 3a_1)x - 2(a_0 + a_1)x^2.$$

We know that the standard basis:

$$S = \{1, x, x^2\} \subseteq P_2[x].$$

$$T(1) = T(1+0\lambda+0\lambda^2) = -(1) + (2)\lambda - 2(1)\lambda^2 \\ = -1 + 2\lambda - 2\lambda^2$$

$$T(\lambda) = T(0+\lambda+0\lambda^2) = -(2) + (3)\lambda - 2(1)\lambda^2 \\ = -2 + 3\lambda - 2\lambda^2$$

$$T(\lambda^2) = T(0+0\lambda+1\lambda^2) = -1 + (0)\lambda - 2(0)\lambda^2 \\ = -1$$

$$\therefore T(1) = -1 + 2\lambda - 2\lambda^2 = -1(1) + 2(\lambda) - 2(\lambda^2)$$

$$T(\lambda) = -2 + 3\lambda - 2\lambda^2 = -2(1) + 3(\lambda) - 2(\lambda^2)$$

$$T(\lambda^2) = -1 = (-1)(1) + 0\lambda + 0\lambda^2$$

$$\therefore B = (T:S) = \begin{bmatrix} -1 & -2 & -1 \\ 2 & 3 & 0 \\ -2 & -2 & 0 \end{bmatrix}$$

which is required matrix of a linear transformation w.r.t standard basis.

Let us find characteristic roots

of B :

$$\text{we } |B - \lambda I| = \begin{vmatrix} -1-\lambda & -2 & -1 \\ 2 & 3-\lambda & 0 \\ -2 & -2 & 0-\lambda \end{vmatrix} \\ \Rightarrow \lambda = 1, 2, -1$$

Let us find characteristic
 vectors corr. to $\lambda = 1$.

corr. to $\lambda = 1$:

We have

$$(A - \lambda I)x = 0$$

$$\Rightarrow (A - I)x = 0$$

$$\Rightarrow \begin{bmatrix} -1 & -1 & -2 & -1 \\ 2 & 3 & -1 & 0 \\ -2 & -2 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & -2 & -1 \\ 2 & 2 & 0 \\ -2 & -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & -2 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{array}{l} R_2 \rightarrow R_2 + R_1 \\ R_3 \rightarrow R_3 - R_1 \end{array}$$

$$\Rightarrow -2x - 2y - z = 0$$

$$-z = 0 \Rightarrow z = 0$$

$$\therefore \boxed{z = -y}$$

$$\therefore x = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} -y \\ y \\ 0 \end{bmatrix} = y \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = yx,$$

where $x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is required
 ch. vector
 corr. to $\lambda = 1$.

corr to $\lambda = -1$:

we have $(A - \lambda I)x = 0$

$$(A + I)x = 0$$

$$\Rightarrow \begin{bmatrix} 0 & -2 & -1 \\ +2 & 4 & 0 \\ -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & -2 & 1 \\ +2 & 4 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\Rightarrow \begin{bmatrix} -2 & -2 & 1 \\ 0 & 2 & +1 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_2 \rightarrow R_2 + R_1$$

$$\Rightarrow \begin{bmatrix} -2 & -2 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3 \rightarrow R_3 + R_2$$

$$R_3 \rightarrow R_3 + 3R_2$$

$$\Rightarrow -2\lambda - 2y + z = 0$$

$$-2y - z = 0$$

$$\Rightarrow \boxed{z = -2y}$$

$$-2\lambda - 2y - 2y = 0$$

$$\Rightarrow -2\lambda - 4y = 0$$

$$\Rightarrow \boxed{\lambda = -2y}$$

$$\therefore X = \begin{bmatrix} \lambda \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2y \\ y \\ -2y \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$$

where $X_2 = \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$ required ch. vector.

corr. to $\lambda = 2$:

$$\therefore (B - 2I)X = 0$$

$$\Rightarrow \begin{bmatrix} -3 & -2 & -1 \\ 2 & 1 & 0 \\ -2 & -2 & -2 \end{bmatrix} \begin{bmatrix} \lambda \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & -2 & -2 \\ 2 & 1 & 0 \\ -3 & -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ -3 & -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_1 \rightarrow \frac{1}{2} R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 + 3R_1 \end{array}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_2 \rightarrow R_2 + R_2$$

$$\Rightarrow x + y + z = 0$$

$$-y - 2z = 0 \Rightarrow \boxed{y = -2z}$$

$$x - 2z + z = 0 \Rightarrow \boxed{x = z}$$

$$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ -2z \\ z \end{bmatrix} = z \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

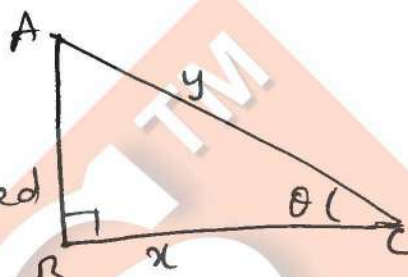
$$\therefore X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = z X_3 \text{ required ch. vector.}$$

215)

If the sum of the lengths of the hypotenuse and another side of a right-angled triangle be given, then find the angle between these sides so that the area of the triangle is maximum.

Soln:

Let ABC be a right-angled triangle in which side BC = x (say) and hypotenuse AC = y (say)



Given $x + y = k$ (constant)

$$\Rightarrow y = k - x$$

Now the area of the triangle ABC is given

$$\text{by } A = \frac{1}{2} BC \cdot AB$$

$$= \frac{1}{2} x \sqrt{y^2 - x^2}$$

$$\therefore AB^2 = y^2 - x^2$$

$$AB = \sqrt{y^2 - x^2}$$

$$= \frac{1}{2} x \sqrt{(k-x)^2 - x^2} = \frac{1}{2} x \sqrt{k^2 - 2kx}$$

$$\text{Let } u = A^2 = \frac{1}{4} x^2 (k^2 - 2kx)$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{2} k (kx - 3x^2) \quad \text{and} \quad \frac{d^2u}{dx^2} = \frac{1}{2} k (k - 6x)$$

for max. or min. of u , $\frac{du}{dx} = 0 \Rightarrow x = k/3$
 ($\because x \neq 0$)

$$\text{When } x = k/3, \quad \frac{d^2u}{dx^2} = \frac{1}{2} k (k - 2k) = -\frac{k^2}{2} < 0$$

$\Rightarrow$ i.e., A , is max when $x = k/3$ and

$$\text{When } y = k - x = 2k/3$$

$$\text{Now } \cos \theta = BC/AC = x/y = \frac{1}{2} \Rightarrow \theta = \pi/3$$

Hence the required angle is $\pi/3$.

2(c). Find the points on the lines:

$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} \text{ and}$$

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1} \text{ which are}$$

nearest to each other.

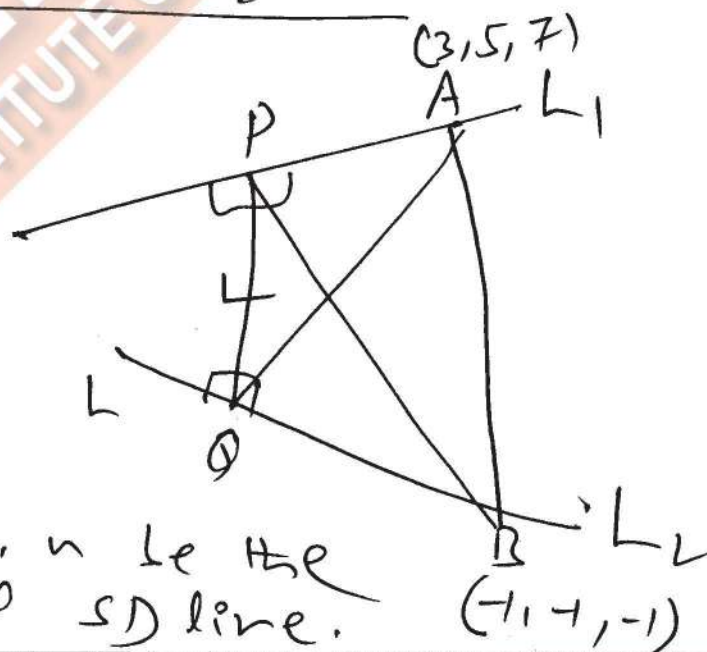
Hence find the shortest distance between the lines and its equation.

Sol Given that

$$L_1: \frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} \text{ and}$$

$$L_2: \frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$$

Let us find S.D b/w L_1 & L_2
 and S.D line equations:



Let l, m, n be the
 dir's of S.D line.

Since SD line is $\perp$ lar to L_1 & L_2

$$\therefore l - 2m + n = 0$$

$$7l - 6m + n = 0$$

$$\frac{l}{-2+6} = \frac{m}{7-1} = \frac{n}{-6+14}$$

$$\Rightarrow \frac{l}{4} = \frac{m}{6} = \frac{n}{8}$$

$\therefore$ The D.O's of SD line are

$$\frac{4}{\sqrt{116}}, \frac{6}{\sqrt{116}}, \frac{8}{\sqrt{116}}$$

$$SD = l(x_2 - x_1) + m(y_2 - y_1) + n(x_2 - x_1)$$

$$= \frac{4}{\sqrt{116}}(4) + \frac{6}{\sqrt{116}}(6) + \frac{8}{\sqrt{116}}(8)$$

$$= \frac{1}{\sqrt{116}}(16 + 36 + 64) = \frac{116}{\sqrt{116}} = \underline{\underline{\sqrt{116}}}$$

Let us find the equations of SD line:

Equation of APQ plane:

$$\begin{vmatrix} x-3 & y-5 & z-7 \\ 1 & -2 & 1 \\ 4 & 6 & 8 \end{vmatrix} = 0$$

$$\Rightarrow (x-3)(-16-6) - (y-5)(8-4) + (z-7)(6+8) = 0$$

$$\Rightarrow -22(x-3) - 4(y-5) + 14(z-7) = 0$$

$$\Rightarrow -22x - 4y + 14z - 12 = 0$$

$$\Rightarrow 22x + 4y - 14z - 12 = 0$$

$$\Rightarrow 11x + 2y - 7z - 6 = 0 \quad \text{--- (I)}$$

Equation of BPQ plane:

$$\begin{vmatrix} x+1 & y+1 & z+1 \\ 7 & -6 & 1 \\ 4 & 6 & 8 \end{vmatrix} = 0$$

$$\Rightarrow (x+1)(-48-6) - (y+1)(56-4) + (z+1)(42+24) = 0$$

$$\Rightarrow (x+1)(-54) - (y+1)(52) + (z+1)(66) = 0$$

$$\Rightarrow (x+1)(-27) - (y+1)(26) + (z+1)(33) = 0$$

$$\Rightarrow 27x + 26y - 33z + 20 = 0 \quad \text{--- (II)}$$

Equations (I) & (II) are together known as S.D line.

Q(5). Given $P_3[x] = \{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_0, a_1, a_2, a_3 \in \mathbb{R}\}$
 as a vector space over the field $\mathbb{R}$

(i) Let $H = \{p(x) \in P_3[x] \mid \int_{-1}^1 p(x) dx = 0\}$.
 prove that H is a subspace of $P_3[x]$ and find a basis of H .

(ii) Find a subspace K of $P_3[x]$
 such that $P_3[x] = H \oplus K$.

sol Given that

$$P_3[x] = \{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

is a vector space.

(i) Let $0(x) = 0 + 0x + 0x^2 + 0x^3 \in P_3[x]$.

$$\therefore \int_{-1}^1 0(x) dx = 0$$

$$\therefore 0(x) \in H.$$

$$\therefore H \neq \emptyset.$$

Let $p(x), q(x) \in H$ then $\int_{-1}^1 p(x) dx = 0$

$$\text{and } \int_{-1}^1 q(x) dx = 0$$

Let $a, b \in \mathbb{R}$

$\therefore a p(x) + b q(x) \in H$

because $\int_{x=-1}^1 [a p(x) + b q(x)] dx$

$$= a \int_{x=-1}^1 p(x) dx + b \int_{x=-1}^1 q(x) dx = 0$$

$$= a(0) + b(0)$$

$$= 0$$

$\therefore H$ is a subspace of $P_3[\mathbb{R}]$

Let $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \in H$

Then $\int_{x=-1}^1 (a_0 + a_1 x + a_2 x^2 + a_3 x^3) dx = 0$

$$\Rightarrow \left[a_0 x + \frac{a_1 x^2}{2} + \frac{a_2 x^3}{3} + \frac{a_3 x^4}{4} \right]_{x=-1}^1 = 0$$

$$\Rightarrow \left(a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{4} \right) - \left(-a_0 + \frac{a_1}{2} - \frac{a_2}{3} + \frac{a_3}{4} \right) = 0$$

$$\Rightarrow \frac{2a_0 + 2a_2}{3} = 0$$

$$\Rightarrow a_0 = -\frac{a_2}{2}$$

$$\therefore H = \left\{ -\frac{a_2}{3} + a_1x + a_2x^2 + a_3x^3 \mid a_1, a_2, a_3 \in \mathbb{R} \right\}$$

$$\therefore H = \left\{ a_1x + a_2\left(\frac{1}{3} + x^2\right) + a_3x^3 \mid a_1, a_2, a_3 \in \mathbb{R} \right\} \subseteq P_3[x]$$

Since there are three free variables say a_1, a_2, a_3

$$\therefore \dim(H) = 3.$$

and basis of H is

$$\left\{ x, \frac{1}{3} + x^2, x^3 \right\}.$$

(ii) Let us find a subspace K of $P_3[x]$ s.t. $P_3[x] = H \oplus K$.

since the dimension of $P_3[x]$ is 4.

$$\text{and } \dim(H) = 3.$$

$$\therefore \dim(K) = 1.$$

and $K = \{1\}$ (constant poly)

check: $\int_0^1 1 dx = 2 \neq 0 \therefore 1 \notin H$

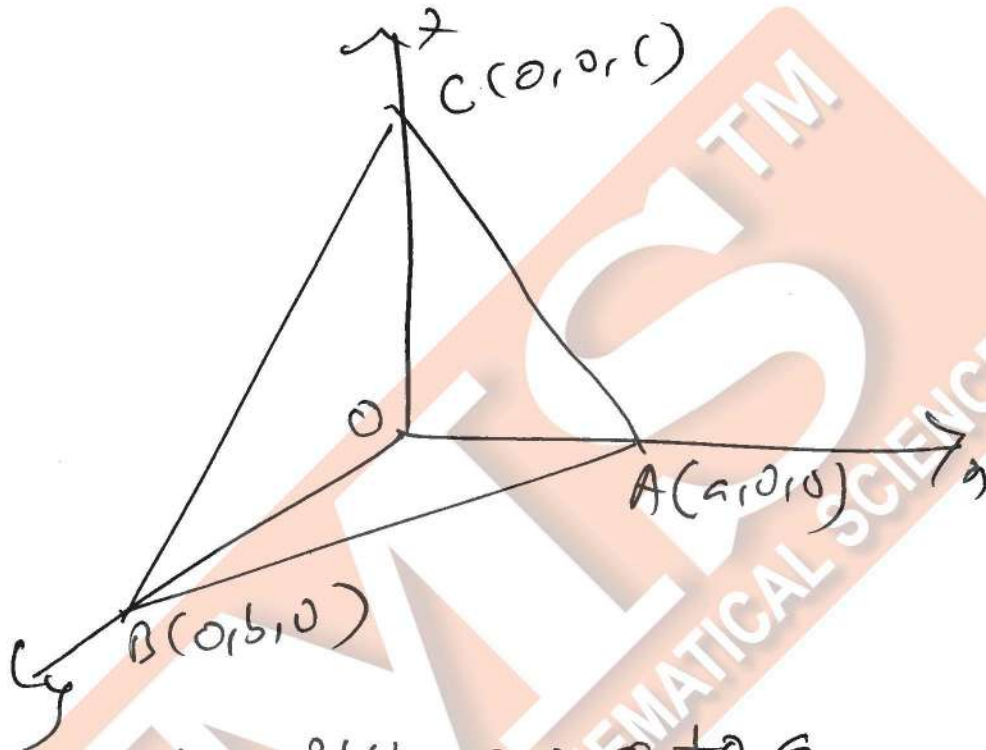
clearly $H \cap K = \{0\}$.

$$\text{Also } \dim(H) + \dim(K) = 3 + 1 = 4 \\ = \dim P_3[A].$$

$$\therefore H \oplus K = P_3[A]$$

36(i) Given that $x=0, y=0, z=0$

$$\text{and } \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$



Limits: $x: 0 \text{ to } a$
 $y: 0 \text{ to } b(1 - \frac{x}{a})$
 $z: 0 \text{ to } c(1 - \frac{x}{a} - \frac{y}{b})$

Let us find the volume enclosed by four faces of the tetrahedron;

$$\therefore \text{Volume} = \int_{x=0}^a \int_{y=0}^{b(1-x)} \int_{z=0}^{c(1-x-\frac{y}{b})} dz dy dx$$

$$= \frac{abc}{6} \quad (\text{Do yourself})$$



3(b)

(i) Evaluate the double integral
 $\int_{x=0}^2 \int_{y=x^2+1}^{2x+1} x^2 y \, dy \, dx$ by reversing
 the order of integration.

(ii) Find the volume of the solid
 which is formed from the
 integration of the coordinate
 planes $x=0, y=0, z=0$ and
 the plane $\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$.

Sol Let us find $\int_{x=0}^2 \int_{y=x^2+1}^{2x+1} x^2 y \, dy \, dx$
 by reversing the order of
 integration.

Given limits: $x=0$ to 2

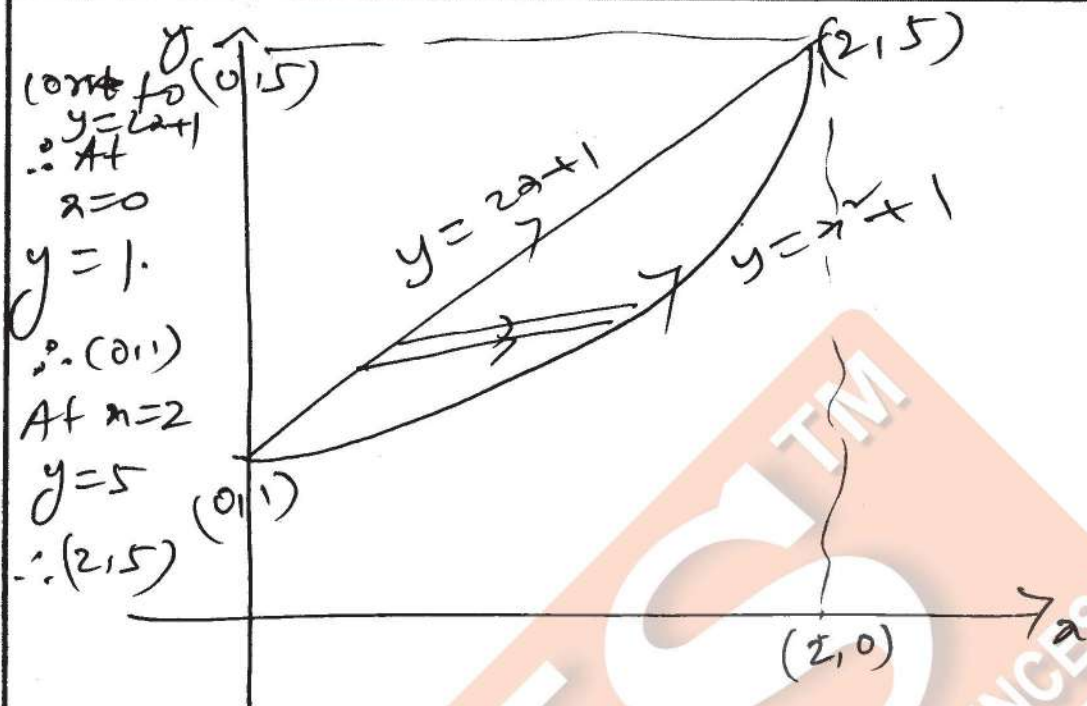
$y=x^2+1$ to $y=2x+1$,

i.e. $0 \leq x \leq 2$ &

$x^2+1 \leq y \leq 2x+1$

we have

$$\begin{aligned} x^2+1 &= 2x+1 \\ \Rightarrow x(x-2) &\Rightarrow x=0, 2. \end{aligned}$$



Given limits are vertical limits
Now let us change vertical limits to horizontal limits.

$\therefore$ y limits: $1 \leq y \leq 5$

x limits: $x = \frac{y-1}{2}$ to $x = +\sqrt{y-1}$

as it is in 1st quadrant

$$\therefore I = \int_{x=0}^{\frac{y-1}{2}} \int_{y=x^2+1}^{2x+1} x^2 y \, dy \, dx = \int_{y=1}^5 \int_{x=\frac{y-1}{2}}^{\sqrt{y-1}} x^2 y \, dx \, dy$$

$y=1$ $x=\frac{y-1}{2}$

$$= \int_{y=1}^5 \left[\int_{x=\frac{y-1}{2}}^{\sqrt{y-1}} 2xy \, dx \right] dy$$

$$= \int_{y=1}^5 \left[\frac{x^2}{3} \right]_{x=\frac{y-1}{2}}^{\sqrt{y-1}} dy = \int_{y=1}^5 \frac{y}{3} \left[(\sqrt{y-1})^3 - \left(\frac{y-1}{2}\right)^3 \right] dy$$

$$= \frac{1}{3} \int_{y=1}^5 \left[(y-1)^{3/2} - \frac{(y-1)^3}{8} \right] dy$$

Let $y-1 = u$

$$\boxed{dy = du}$$

$\therefore u : 0 \text{ to } 4$

$$\therefore I = \frac{1}{3} \int_{u=0}^4 (u+1) \left(u^{3/2} - \frac{u^3}{8} \right) du$$

$$= \frac{184}{35} \quad (\text{please complete it})$$

3.(c) → If $x/1 = y/2 = z/3$ represent one of a set of three mutually $\perp$ lar generators of the Cone $5y^2 - 8zx - 3xy = 0$, find the equations of the other two.

Solⁿ: If $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ is one of the three mutually $\perp$ lar generators, then it is normal to the plane through the vertex cutting the Cone in two $\perp$ lar generators and therefore the equation of the plane is

$$x + 2y + 3z = 0 \quad \text{--- (1)}$$

Now we are to find the lines of intersection of this plane and the given cone and let one of the lines of intersection be

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

Then we have $l + 2m + 3n = 0$ and $5mn - 8nl - 3lm = 0$ --- (2)

Eliminating l between these we get

$$5mn - (8n + 3m)[- (2m + 3n)] = 0$$

$$\Rightarrow 24n^2 + 30mn + 6m^2 = 0$$

$$\Rightarrow m^2 + 5mn + 4n^2 = 0$$

$$\Rightarrow (m+n)(m+4n) = 0$$

When $m = -n$, from (2) we get $l + n = 0 \Rightarrow l = -n$

$$\therefore \frac{l}{1} = \frac{m}{1} = \frac{n}{(-1)} \quad \text{--- (3)}$$

When $m = -4n$ from (2) we get $l - 5n = 0 \Rightarrow l = 5n$

$$\therefore \frac{l}{5} = \frac{m}{(-4)} = \frac{n}{1} \quad \text{--- (4)}$$

Hence from (3) and (4) the other two generators

$$\text{are } \frac{x}{1} = \frac{y}{1} = \frac{z}{(-1)} \quad \text{and} \quad \frac{x}{5} = \frac{y}{-4} = \frac{z}{1}$$

and evidently these are $\perp$ lar as $1 \cdot 5 + 1(-4) + (-1)1 = 0$ and also each one of them is $\perp$ lar to the given generator.

4(c)

Let F be a subfield of complex numbers and let T be the linear transformation from F^3 to F^3 defined by

$$T(x, y, z) = (x - y + 2z, 2x + y, -x - 2y + 2z)$$

If (a, b, c) is a vector in F^3 then answer the following.

(i) find the condition on a, b, c such that the vector (a, b, c) be in the null space of T . What is the nullity of T ?

(ii) find the condition on a, b, c such that the vector (a, b, c) be in the range of T . What is the rank of T ?

Sol Given Linear transformation
 $T: F^3 \rightarrow F^3$ s.t

$$T(x, y, z) = (x - y + 2z, 2x + y, -x - 2y + 2z)$$

(i)
 Let $N(T) = \{(x, y, z) \in F^3 / T(x, y, z) = (0, 0, 0)\}$
 Let $(a, b, c) \in N(T)$ be in null space of T .

$$\text{Then } T(a, b, c) = (0, 0, 0)$$

$$\Rightarrow \begin{aligned} a - b + 2c &= 0 \Rightarrow a + 2a + 2c = 0 \\ 2a + b &= 0 \Rightarrow \boxed{b = -2a} \Rightarrow 2c = -3a \\ -a - 2b + 2c &= 0 \Rightarrow c = -\frac{3}{2}a \end{aligned}$$

$$\therefore \boxed{\begin{aligned} b &= -2a \\ c &= -\frac{3}{2}a \end{aligned}}$$

$$\therefore N(T) = \left\{ (a, -2a, -\frac{3}{2}a) / a \in F \right\}$$

Since there is only free variable a ,

$$\therefore \dim(N(T)) = 1 = \text{nullity of } T.$$

$$\therefore \text{Basis of } N(T) \text{ is } \left\{ (1, -2, -\frac{3}{2}) \right\}$$

$$(ii) \text{ Let } R(T) = \left\{ T(x, y, z) / x, y, z \in F \right\}$$

be a range space of T .

$$\therefore R(T) = \left\{ (x - y + 2z, 2x + y, -x - 2y + 2z) / x, y, z \in F \right\}$$

$$\text{Let } (a, b, c) \in R(T).$$

$$\text{Then } (a, b, c) = (x - y + 2z, 2x + y, -x - 2y + 2z)$$

$$= \lambda$$

Let $(x-y+2z, 2x+y, -x-2y+2z) \in R(T)$
 then $(x-y+2z, 2x+y, -x-2y+2z)$
 $= x(1, 2, -1) + y(-1, 1, -2) + z(2, 0, 2)$

we have

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & -2 \\ 2 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -3 \\ 0 & -4 & 4 \end{bmatrix} \begin{array}{l} R_2 \rightarrow R_2 + R_1 \\ R_3 \rightarrow R_3 - 2R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{array}{l} R_2 \rightarrow \frac{1}{3}R_2 \\ R_3 \rightarrow \frac{1}{4}R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} R_3 \rightarrow R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} R_1 \rightarrow R_1 - 2R_2$$

clearly it is in row-reduced echelon form

$$\therefore R(T) = \{ x(1, 0, 1) + y(0, 1, -1) / x, y \in F \}$$

$$= \{ (x, y, x-y) / x, y \in F \}$$

Let $(a, b, c) \in R(T)$

Then $(a, b, c) = (x, y, x-y)$

$$\Rightarrow a = x$$

$$b = y$$

$$c = x - y$$

Since there are only two free variables say x & y

$$\therefore \dim(R(T)) = 2 = \text{rank}(T)$$

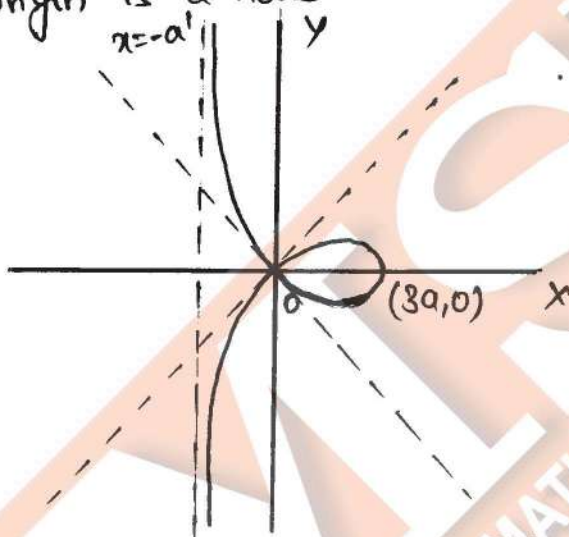
4.(b)

Trace $y^2(a+x) = x^2(3a-x)$

Solution: Symmetry. The curve is symmetrical about the x -axis.

Origin. The curve passes through the origin. The tangents at the origin are $y^2 - 3x^2 = 0 \Rightarrow y = \pm\sqrt{3}x$.

So the origin is a node on the curve.



Points of Intersection. Put $y=0$, we get $x=3a$. Thus the given curve passes through the point $(3a, 0)$. The curve intersects y -axis at the origin.

Asymptote. The asymptote parallel to the y -axis is $x = -a$.

4(c)(i), Find the equations to the generating lines of the hyperboloid $\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{16} = 1$ which pass through the points $(2, 3, -4)$ and $(2, -1, 4/3)$.

Solⁿ: Any line through $(2, 3, -4)$ and $(2, -1, 4/3)$

$$\text{is } \frac{x-2}{1} = \frac{y-3}{m} = \frac{z+4}{n} = r \text{ (say)} \quad \text{--- (1)}$$

$\therefore$ Any point on this line is

$(lr+2, mr+3, nr-4)$ and it lies on the given hyperboloid if

$$\frac{(lr+2)^2}{4} + \frac{(mr+3)^2}{9} - \frac{(nr-4)^2}{16} = 1$$

$$r^2 \left(\frac{l^2}{4} + \frac{m^2}{9} - \frac{n^2}{16} \right) + 2r \left(\frac{2l}{4} + \frac{3m}{9} + \frac{4n}{16} \right) = 0 \quad \text{--- (2)}$$

If the line (1) is a generator of the given hyperboloid, then (1) lies wholly on the hyperboloid and the conditions for which from (2) are

$$\frac{l^2}{4} + \frac{m^2}{9} - \frac{n^2}{16} = 0 \quad \text{and} \quad \frac{2l}{4} + \frac{3m}{9} + \frac{4n}{16} = 0$$

$$\Rightarrow \frac{l^2}{4} + \frac{m^2}{9} - \frac{n^2}{16} = 0 \quad \text{and} \quad \frac{l}{2} + \frac{m}{3} = -\frac{n}{4} \quad \text{--- (3)}$$

Eliminating n , we get

$$\frac{l^2}{4} + \frac{m^2}{9} - \left(\frac{l}{2} + \frac{m}{3} \right)^2 = 0$$

$$\Rightarrow -\frac{1}{3}lm = 0$$

$$\Rightarrow \text{either } l = 0 \text{ or } m = 0$$

when $l = 0$, from (3), we get $\frac{m}{3} = -\frac{n}{4}$

when $m = 0$, from (3), we get $\frac{l}{2} = -\frac{n}{4}$
 $\Rightarrow \frac{l}{1} = -\frac{n}{2}$

Hence from ①, Equations of the
 required generator through $(2, 3, -4)$

are $\frac{x-2}{0} = \frac{y-3}{3} = \frac{z+4}{-4}$

and $\frac{x-2}{1} = \frac{y-3}{0} = \frac{z+4}{-2}$

Similarly, we can find the generators
 through the point $(2, -1, 4/3)$ are

$\frac{x-2}{0} = \frac{y+1}{3} = \frac{z-(4/3)}{-4}$

and $\frac{x-2}{2} = \frac{y+1}{6} = \frac{z-(4/3)}{10}$

=====

4(c) (i) Find the equation of the right circular cylinder whose axis is $x=2y=-z$, and radius is 4. Plot the area of the section of this cylinder by the plane $z=0$ is 24π .

Sol

Given axis is
 $x=2y=-z$

$$\Rightarrow \frac{x}{2} = \frac{y}{1} = \frac{z}{-2} \quad \text{--- (1) and radius is '4'}$$



D.C's of axis line is $\frac{x}{2} = \frac{y}{1} = \frac{z}{-2}$

$$\frac{2}{\sqrt{9}}, \frac{1}{\sqrt{9}}, \frac{-2}{\sqrt{9}} \Rightarrow \frac{2}{3}, \frac{1}{3}, \frac{-2}{3}$$

$$\begin{aligned} \therefore OQ &= \text{projection of } OP \text{ along axis line.} \\ &= \frac{2}{3}(x-0) + \frac{1}{3}(y-0) + \left(\frac{-2}{3}\right)(z-0) \\ &= \frac{1}{3}[2x + y - 2z]. \end{aligned}$$

$$\text{We know that } (OP)^2 = (OQ)^2 + (PQ)^2$$

$$\Rightarrow x^2 + y^2 + z^2 = \left[\frac{1}{3} (2x + y - 2z) \right]^2 + 16$$

$$\Rightarrow x^2 + y^2 + z^2 = \frac{1}{9} (4x^2 + y^2 + 4z^2 + 4xy - 8xz - 4yz) + 16$$

$$\Rightarrow 9x^2 + 9y^2 + 9z^2 = 4x^2 + y^2 + 4z^2 + 4xy - 8xz - 4yz + 144$$

$$\Rightarrow 5x^2 + 8y^2 + 5z^2 - 4xy + 8xz + 4yz = 144$$

which is required right circular cylinder.

Since the area of the section by $z=0$.

$$\therefore 5x^2 + 8y^2 - 4xy = 144$$

clearly it is an ellipse in the plane $z=0$.

Its area is given by

$$\frac{\pi a b}{\sin \theta}$$

where $r=4$ is the radius of cylinder.

and β is the angle b/w the axis line and cutting plane $z=0$.

— The axis line d.o.s are $\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}$.

— The angle α b/w the axis and the normal to $z=0$.

which is in the

direction of $\hat{k}$.

$$\therefore \cos \alpha = -\frac{2}{3} \Rightarrow |\cos \alpha| = \frac{2}{3}$$

Since the angle b/w the axis and the plane itself is $\beta = 90^\circ - \alpha$

$$\therefore \sin \beta = \sin(90^\circ - \alpha) = \cos \alpha = \frac{2}{3}$$

$$\therefore \text{Area} = \frac{\pi r^2}{\sin \beta} = \frac{\pi (16)}{\frac{2}{3}} = 24\pi$$

==

Section - B

5(c) If y_1 and y_2 are two solutions of $\frac{dy}{dx} + p(x)y = Q(x)$ and

$y_2 = y_1 u$ then show that

$$u = 1 + c e^{-\int \left(\frac{Q}{y_1}\right) dx}$$

where 'c' is non-zero constant.

sol Given that $\frac{dy}{dx} + p(x)y = Q(x)$

Given that y_1 and y_2 are two solutions of (1).

$$\therefore \frac{dy_1}{dx} + p y_1 = Q \quad \text{--- (2)}$$

$$\frac{dy_2}{dx} + p y_2 = Q \quad \text{--- (3)}$$

Given that $y_2 = y_1 u$

$$\therefore \frac{dy_2}{dx} = y_1 \frac{du}{dx} + u \frac{dy_1}{dx}$$

$$\therefore (3) \equiv y_1 \frac{du}{dx} + u \frac{dy_1}{dx} + p(y_1 u) = Q.$$

$$\Rightarrow y_1 \frac{dy_1}{dx} + u [\phi - 1' y_1] + p(u y_1) = \phi.$$

$$\Rightarrow y_1 \left[\frac{dy_1}{dx} - p u + p u \right] = \phi (1-u)$$

$$\Rightarrow y_1 \left[\frac{dy_1}{dx} \right] = \phi (1-u).$$

$$\Rightarrow \frac{dy_1}{1-u} = \frac{\phi}{y_1} dx$$

$$\Rightarrow -\log(1-u) = \int \frac{\phi}{y_1} dx + C$$

$$\Rightarrow \log(1-u) = -\int \frac{\phi}{y_1} dx - C$$

$$\Rightarrow 1-u = e^{-\int \frac{\phi}{y_1} dx - C}$$

$$= e^{-\int \frac{\phi}{y_1} dx} \cdot a_1 \quad (\because a_1 = e^{-C})$$

$$\Rightarrow u = 1 - a_1 e^{-\int \frac{\phi}{y_1} dx}$$

$$\Rightarrow u = 1 + a e^{-\int \frac{\phi}{y_1} dx}$$

which is required
result

($\because a_1$ is an
 arbitrary
 const.)

54) Apply Laplace transform as
 to solve $(D^2 + D)y(t) = 2$: $D = \frac{d}{dt}$
 s.c. $y(0) = 3, y'(0) = 1$

sol Given that

$$(D^2 + D)y(t) = 2 \quad ; \quad D = \frac{d}{dt}$$

$$\Rightarrow y''(t) + y'(t) = 2$$

Apply in g L.T,

$$L\{y''(t) + y'(t)\} = L\{2\}$$

$$\Rightarrow L\{y''(t)\} + L\{y'(t)\} = L\{2\}$$

$$\Rightarrow p^2 L\{y(t)\} - p\{y(0)\} - y'(0) + p L\{y(t)\} - y(0) = L\{2\}$$

$$\Rightarrow p^2 L\{y(t)\} - p(3) - (1) + p L\{y(t)\} - 3 = L\{2\}$$

$$\Rightarrow L\{y(t)\} (p^2 + p) - 3p - 4 = \frac{2}{p}$$

$$\Rightarrow L\{y(t)\} (p^2 + p) = \frac{2}{p} + 3p + 4$$

$$\Rightarrow L\{y(t)\} = \frac{2 + 3p^2 + 4p}{p(p^2 + p)}$$

$$\therefore \mathcal{L}\{y''\} = \frac{3p^2 + 4p + 2}{p^2(p+1)}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1} \left\{ \frac{3p^2 + 4p + 2}{p^2(p+1)} \right\}$$

We have

$$\frac{3p^2 + 4p + 2}{p^2(p+1)} = \frac{A}{p} + \frac{B}{p^2} + \frac{C}{p+1} \quad \text{--- (I)}$$

$$\Rightarrow 3p^2 + 4p + 2 =$$

$$A(p)(p+1) + B(p+1) + Cp^2$$

$$\text{If } p=0$$

then

$$2 = B(1)$$

$$\therefore \boxed{B=2}$$

$$\text{If } p=-1$$

then

$$3 - 4 + 2 = C$$

$$\Rightarrow \boxed{C=1}$$

$$\text{If } p=1$$

then

$$3 + 4 + 2 = A(2) + B(2) + C$$

$$\Rightarrow 9 = 2A + 4 + 1$$

$$\Rightarrow 4 = 2A$$

$$\Rightarrow \boxed{A=2}$$

$$\therefore \frac{3p^2 + 4p + 2}{p^2(p+1)} = \frac{2}{p} + \frac{2}{p^2} + \frac{1}{p+1}$$

$$\therefore y(t) = L^{-1} \left\{ \frac{2}{p} \right\} + L^{-1} \left\{ \frac{2}{p^2} \right\} + L^{-1} \left\{ \frac{1}{p+1} \right\}$$

$$\boxed{y(t) = 2(1) + 2(t) + e^{-t}}$$

which is required solution

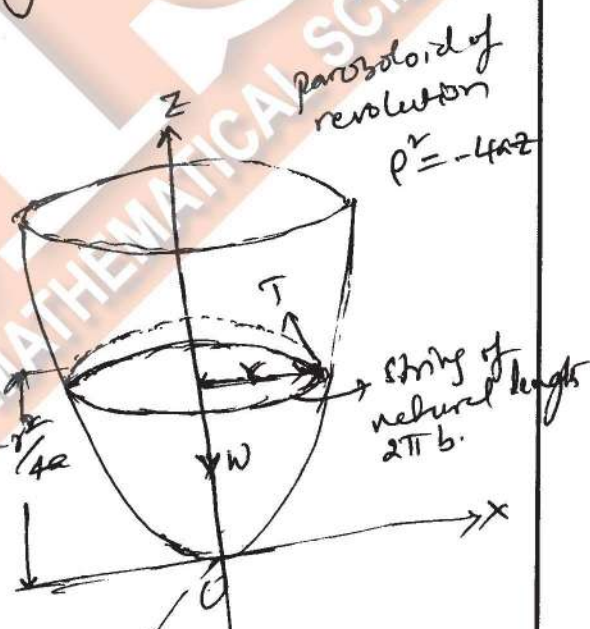
5(c) A smooth paraboloid of revolution is fixed with its axis vertical and vertex upwards; on it is placed a heavy elastic string of unstretched length $2\pi b$. When the string is in equilibrium, apply the principle of virtual work to show that it rests in the form of a circle of radius $\frac{4\pi ab\lambda}{4\pi a\lambda - wb}$, where w is the weight of the string, λ is the modulus of elasticity and $4a$ is the latus rectum of the generating parabola.

Sol.

Let the string settle in the form of a circle of radius r on the paraboloid. The paraboloid has latus rectum $4a$, so its equation can be written as

$$\rho^2 = -4az$$

Since its vertex is upwards. Hence, for a circular ring of radius r ,

$$z = -\frac{r^2}{4a}$$


In equilibrium it becomes a circle of radius r

$$\therefore \frac{dz}{dr} = \frac{-r}{2a}$$

The natural length of the string is $l_0 = 2\pi b$
while its actual length (i.e. stretched length of
radius r) is $l = 2\pi r$

Thus its extension is $= l - l_0 = 2\pi(r - b)$

By Hooke's law,

If T is the tension, using the modulus of
elasticity λ ,

$$T = \frac{\lambda(l - l_0)}{l_0} = \frac{\lambda 2\pi(r - b)}{2\pi b} = \frac{\lambda(r - b)}{b} \quad \text{--- (1)}$$

Give the ring a small virtual displacement dr .
work done by tension (T):

$$\delta W_T = -T d(2\pi r) = -2\pi T dr$$

work done by the weight (W):

$$\text{Since } dz = \frac{-r}{2a} dr$$

$$\delta W_W = -W \delta z = \frac{Wr}{2a} dr$$

for equilibrium, total work done is zero

$$\therefore -2\pi T dr + \frac{Wr}{2a} dr = 0$$

$$\Rightarrow 2\pi T = \frac{Wr}{2a} \quad \text{--- (2)}$$

from (1) & (2),

$$2\pi \frac{\lambda(r - b)}{b} = \frac{Wr}{2a}$$

$$\Rightarrow 4\pi a \lambda (r - b) = Wb$$

$$\Rightarrow (4\pi a \lambda - Wb) r = 4\pi ab \lambda$$

$$\Rightarrow r = \frac{4\pi ab \lambda}{4\pi a \lambda - Wb}$$

which is the
required radius.

5(d)

A particle falls from rest at the vertex of an inverted catenary. prove that the particle will leave the curve when the path described is $\sqrt{2}$ times the vertical distance through which the particle has fallen.

Soln:

Take the vertex O (top of inverted catenary) as origin.

h - is the vertical drop from O to P.

s - is the arc length travelled along the curve from O to P.

for a catenary,

$$\text{we have } s = c \tanh \psi$$

$$\rho = \frac{ds}{d\psi} = c \sec^2 \psi$$

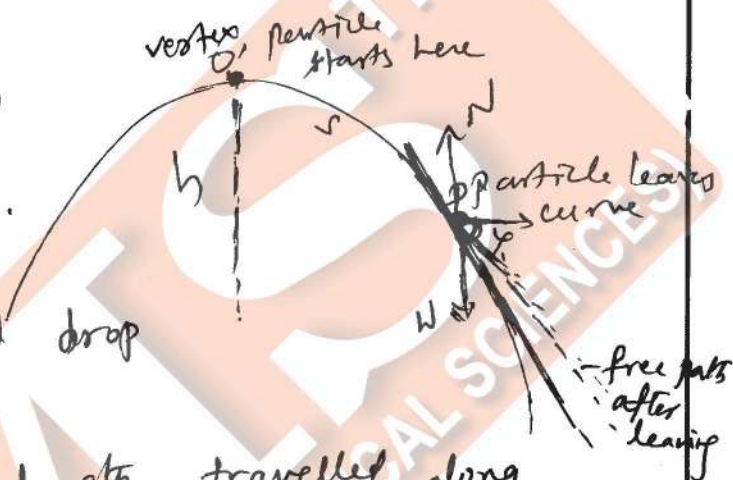
where c is the parameter of the catenary and ρ is the radius of curvature.

$$y = c \sec \psi$$

$$h = y - c = c \sec \psi - c = c(\sec \psi - 1)$$

velocity of the particle at a height h .

$$v^2 = 2gh = 2gc(\sec \psi - 1) \quad \text{--- (1)}$$



condition for leaving the curve.
 The particle rides on the convex (outer) side of the arch, so the curve can only push outward on it. Resolving along the inward normal $W - N = \frac{mv^2}{r}$.

$$mg \cos \phi - N = \frac{mv^2}{r}$$

The particle leaves the curve when $N=0$

$$g \cos \phi = \frac{v^2}{r} \Rightarrow v^2 = gr \cos \phi$$

$$= g(c \sec \phi) \cos \phi$$

$$v^2 = gc \sec \phi$$

from (1) & (2)

$$gc \sec \phi = 2gc (\sec \phi - 1)$$

$$\sec \phi = 2 \sec \phi - 2$$

$$\sec \phi = 2$$

$$\Rightarrow \phi = 60^\circ$$

Comparing arc length and height fallen.
 At this point

$$s = c \tan \phi = c \tan 60 = c\sqrt{3}$$

$$h = c (\sec \phi - 1) = c(2 - 1) = c$$

$$\therefore s = \sqrt{3}c = \sqrt{3}h.$$

$\therefore$ The arc length traversed is $\sqrt{3}$ times the vertical distance fallen

5(p) Given $\vec{F} = (2x+3y)\mathbf{i} - 4z\mathbf{j} - 5x\mathbf{k}$
 and 'S' is the surface

$$2x+4y+4z=5 \text{ bounded}$$

by $x=0, x=2, y=0$ and $y=3$

evaluate $\iint_S (\nabla \times \vec{F}) \cdot \vec{n} dS$.

Sol Given that 'S' is the
 surface $2x+4y+4z=5$
 bounded by $x=0, x=2,$
 $y=0$ and $y=3$.

We have

$$\begin{aligned} \nabla \times \vec{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+3y & -4z & -5x \end{vmatrix} \\ &= \mathbf{i} (+4) - \mathbf{j} (0-5) + \mathbf{k} (0-3) \\ &= +4\mathbf{i} + 5\mathbf{j} - 3\mathbf{k} \end{aligned}$$

Let $S: \phi = 2x+4y+4z-5=0$

$$\therefore \nabla \phi = 2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$$

$$\therefore \hat{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{2i + 4j + 4k}{\sqrt{4 + 16 + 16}}$$

$$= \frac{2i + 4j + 4k}{6}$$

$$= \frac{1}{3}i + \frac{2}{3}j + \frac{2}{3}k$$

$$\therefore (\nabla \times \vec{F}) \cdot \hat{n} = 4\left(\frac{1}{3}\right) + 5\left(\frac{2}{3}\right) - 3\left(\frac{2}{3}\right)$$

$$= \frac{4}{3} + \frac{10}{3} - \frac{6}{3}$$

$$= \frac{8}{3}$$

$$\therefore ds = \frac{dxdy}{|\hat{n} \cdot \vec{k}|} = \frac{dxdy}{\frac{2}{3}}$$

$$\therefore \iint_S (\nabla \times \vec{F}) \cdot \hat{n} ds = \iint_R \frac{8}{3} \frac{dxdy}{\frac{2}{3}}$$

$$= 4 \iint_R dxdy = 4 \times 6$$

$$R: [0, 2] \times [0, 3] = \underline{\underline{24}}$$

where R is rectangle
 and its area : 2×3
 $= 6$

6(a) Solve $x^2 \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} + y = \frac{\sin(\log x) + 1}{x} \log x$

Solⁿ: Given that

$$x^2 \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} + y = \frac{\sin(\log x) + 1}{x} \log x \quad \text{--- (1)}$$

Equation (1) can be written as

$$(x^2 D^2 - 3xD + 1)y = x^{-1} \log x [\sin(\log x) + 1] \quad \text{--- (2)}$$

Let $x = e^z$ so that $z = \log x$ and let $D_1 = \frac{d}{dz}$ --- (3)

Then $x D = D_1$ and $x^2 D^2 = D_1(D_1 - 1)$ --- (4)

Using (3) and (4), (2) reduces to

$$[D_1(D_1 - 1) - 3D_1 + 1]y = e^{-z} z (\sin z + 1)$$

$$\Rightarrow (D_1^2 - 4D_1 + 1)y = e^{-z} z \sin z + e^{-z} z \quad \text{--- (5)}$$

A.E. of (5) is $D_1^2 - 4D_1 + 1 = 0$

$$\Rightarrow D_1 = 2 \pm \sqrt{3}$$

$$\therefore \text{C.F.} = e^{2z} [C_1 \cosh(\sqrt{3}z) + C_2 \sinh(\sqrt{3}z)]$$

$$y_c = x^2 [C_1 \cosh(\sqrt{3} \log x) + C_2 \sinh(\sqrt{3} \log x)]$$

and P.I. corresponding to $e^{-z} z \sin z$

$$= \frac{1}{D_1^2 - 4D_1 + 1} e^{-z} (z \sin z)$$

$$= e^{-z} \frac{1}{(D_1^2 - 1)^2 - 4(D_1 - 1) + 1} z \sin z$$

$$= e^{-z} \frac{1}{D_1^2 - 6D_1 + 6} z \sin z$$

$$\begin{aligned}
 &= e^{-z} \left[\frac{1}{D_1^2 - 6D_1 + 6} \sin z - \frac{2D_1 - 6}{(D_1^2 - 6D_1 + 6)^2} \sin z \right] \\
 &= e^{-z} \left[z \frac{1}{-1 - 6D_1 + 6} \sin z - (2D_1 - 6) \frac{1}{(-1 - 6D_1 + 6)^2} \sin z \right] \\
 &= e^{-z} \left[z \frac{1}{5 - 6D_1} \sin z - \frac{2D_1 - 6}{(5 - 6D_1)^2} \sin z \right] \\
 &= e^{-z} \left[z \frac{5 + 6D_1}{25 - 60D_1 + 36D_1^2} \sin z - (2D_1 - 6) \frac{1}{25 - 60D_1 + 36D_1^2} \sin z \right] \\
 &= e^{-z} \left[z (5 + 6D_1) \frac{1}{25 + 36} \sin z - (2D_1 - 6) \frac{1}{25 - 60D_1 + 36} \sin z \right] \\
 &= e^{-z} \left[\frac{z}{61} (5 + 6D_1) \sin z + (2D_1 - 6) \frac{1}{11 + 60D_1} \sin z \right] \\
 &= e^{-z} \left[\frac{z}{61} (5 \sin z + 6 \cos z) + \frac{(2D_1 - 6)(60D_1 - 11) \sin z}{3600D_1^2 - 121} \right] \\
 &= e^{-z} \left[\frac{z}{61} (5 \sin z + 6 \cos z) + \frac{120D_1^2 - 382D_1 + 66 \sin z}{-3600 - 121} \right] \\
 &= e^{-z} \left[\frac{z}{61} (5 \sin z + 6 \cos z) + \frac{120(-\sin z) - 382 \cos z + 66 \sin z}{-3721} \right] \\
 &= e^{-z} \left[\frac{z}{61} (5 \sin z + 6 \cos z) + \frac{54 \sin z + 382 \cos z}{3721} \right] \\
 &= \frac{1}{x} \left[\frac{\log x}{61} (5 \sin \log x + 6 \cos \log x) + \frac{54 \sin(\log x) + 382 \cos(\log x)}{3721} \right]
 \end{aligned}$$

we have

$$\frac{1}{D^2 - 4D + 1} e^{-x} \cdot x = e^{-x} \frac{1}{(D-1)^2 - 4(D-1) + 1} x$$

$$= e^{-x} \cdot \frac{1}{D^2 + 1 - 2D - 4D + 4 + 1} x$$

$$= e^{-x} \cdot \frac{1}{D^2 - 6D + 6} x$$

$$= \frac{e^{-x}}{6} \left(1 + \left(\frac{D^2 - 6D}{6} \right) \right)^{-1} x$$

$$= \frac{e^{-x}}{6} \left[1 - \left(\frac{D^2 - 6D}{6} \right) + \dots \right] x$$

$$= \frac{e^{-x}}{6} \left[x - \left(-\frac{6}{6} \right) \right]$$

$$= \frac{e^{-x}}{6} [x + 1] = \frac{e^{-\log x}}{6} (\log x + 1)$$

$$= \frac{1}{6x} (\log x + 1)$$

$\therefore y_p(x) = \frac{1}{x} \left[\frac{\log x}{6} (5 \sin \log x + 6 \cos \log x) + 5 \frac{4 \sin(\log x) + 3 \cos(\log x)}{3721} \right] + \frac{1}{6x} (\log x + 1)$

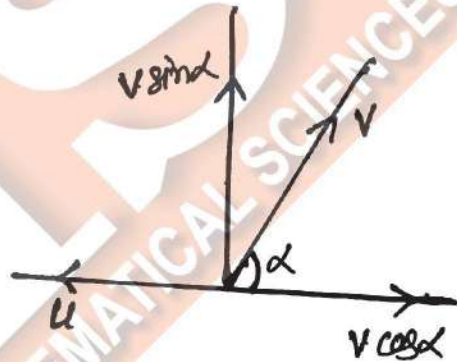
$\therefore$ the gen sol of (1) is given by
 $y = y_c + y_p$ (write your self)

6.(b)(i) → A battleship is steaming ahead with velocity u . A gun is mounted on the ship so as to point straight backwards and is set at an angle of elevation α . If v be the velocity of projection relative to the gun, show that the range is $(2v/g) \sin \alpha (v \cos \alpha - u)$, and the angle of elevation for maximum range is $\cos^{-1} \left[\frac{u + \sqrt{(u^2 + 8v^2)}}{4v} \right]$.

Solⁿ: Since the ship is moving horizontally with a velocity u in a direction opposite that of the projection, therefore the initial actual horizontal velocity of the shot

$$= v \cos \alpha - u.$$

Also the initial actual vertical velocity of the shot $= v \sin \alpha$.



$$\therefore \text{the range } R = \frac{2}{g} (\text{horizontal vel.}) (\text{initial vertical vel.})$$

$$= \frac{2}{g} (v \cos \alpha - u) v \sin \alpha$$

$$= \frac{2v}{g} \sin \alpha (v \cos \alpha - u)$$

Now R is a function of α . So R will be maximum when $dR/d\alpha = 0$.

$$\text{i.e., when } \frac{2v}{g} (v \cos^2 \alpha - v \sin^2 \alpha - u \cos \alpha) = 0.$$

i.e., when $2v \cos^2 \alpha - u \cos \alpha - v = 0$

i.e., when $\cos \alpha = \frac{u \pm \sqrt{(u^2 + 8v^2)}}{4v}$,

The negative sign before the radical is not admissible because it makes the value of $\cos \alpha$ negative or α obtuse.

$\therefore$ the angle of elevation for maximum range is

$$\cos^{-1} \left[\frac{u + \sqrt{(u^2 + 8v^2)}}{4v} \right].$$

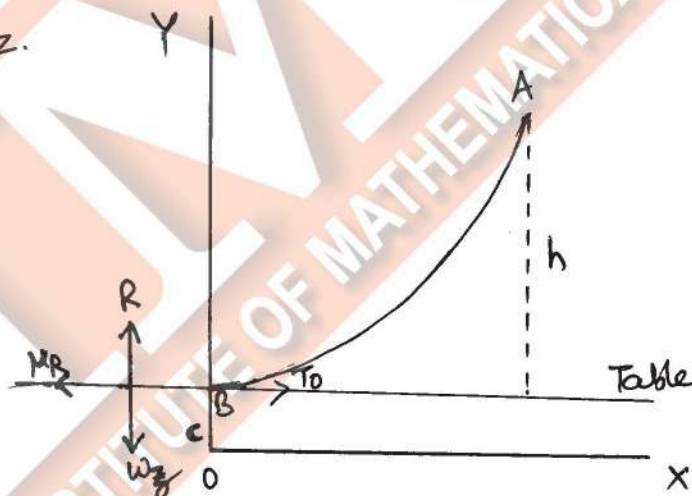
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6(b)ii

A uniform chain of length l has one end fixed at a height h above a rough table and rests in a vertical plane so that a portion of it lies in a straight line on the table. Prove that if the chain is on the point of slipping, the length on the table is $(l + \mu h) - \sqrt{(\mu^2 + 1)h^2 + 2\mu l h}$, where μ is the coefficient of friction.

Solⁿ: Let one end of a uniform chain ABC of length l be fixed at A a height h above the rough table and let the portion BC of the chain rest on the table.

Let $BC = z$.



The chain rests in limiting equilibrium with the portion AB in the form of an arc of a catenary with B as its vertex i.e. the lowest point. Let us consider the limiting equilibrium of the portion BC of the chain lying on the table.

The forces acting on the portion BC of the chain are (i) its weight wz acting vertically downwards, (ii) the normal reaction R of the table acting perpendicular to the table, (iii) the force of limiting friction μR along the table in the direction BC which is opposite to the direction CB in which the chain BC will begin to slip, and (iv) the tension T_0 in the chain at B.

For the equilibrium of the chain BC, resolving these forces horizontally and vertically, we have

$$R = wz \quad \text{and} \quad \mu R = T = wc$$

$$\therefore \mu wz = wc, \text{ (or) } c = \mu z$$

Length of the arc $AB = s_A = l - z$.

and the ordinate of the point $A = y_A = c + h$

$\therefore$ using the formula $y^2 = c^2 + s^2$ for the point A, we have

$$y_A^2 = c^2 + s_A^2$$

$$\Rightarrow (c+h)^2 = c^2 + (l-z)^2$$

$$\Rightarrow z^2 - 2lz + (l^2 - h^2) - 2ch = 0$$

$$\Rightarrow z^2 - 2(l + \mu h)z + (l^2 - h^2) = 0 \quad [\because c = \mu z]$$

$$\therefore z = \frac{2(l + \mu h) \pm \sqrt{4(l + \mu h)^2 - 4(l^2 - h^2)}}{2.1}$$

$$\Rightarrow z = (l + \mu h) \pm \sqrt{(l + \mu h)^2 - (l^2 - h^2)}$$

The plus sign will give $z > l$, which is impossible. Therefore, neglecting the plus sign, the required length z of the chain on the table is given by

$$z = (l + \mu h) - \sqrt{(l + \mu h)^2 - (l^2 - h^2)}$$

$$= (l + \mu h) - \sqrt{(\mu^2 + 1)h^2 + 2\mu lh}$$

6(c). Determine T, N, B, K and τ for the parabola $y^2 = 4ax$, where 'a' is a constant.
Here T, N, B, K and τ denote unit tangent vector, unit principal normal vector, unit binormal vector, curvature and torsion respectively.

Solⁿ: Given curve $y^2 = 4ax$
Let $x = at^2$; $y = 2at$; say t is a parameter.
Clearly it satisfies (1) (2)

Let $\vec{r} = x\vec{i} + y\vec{j}$ — (3)
be a position vector.

then $\vec{r}(t) = at^2\vec{i} + 2at\vec{j}$ — (4)

$$\therefore \vec{r}'(t) = \frac{d\vec{r}}{dt} = 2at\vec{i} + 2a\vec{j}$$

$$\therefore \frac{ds}{dt} = \left| \frac{d\vec{r}}{dt} \right| = \sqrt{4a^2t^2 + 4a^2} \quad (5)$$

$$\begin{aligned} \therefore T = \frac{d\vec{r}}{ds} &= \frac{\frac{d\vec{r}}{dt}}{\left| \frac{d\vec{r}}{dt} \right|} = \frac{2at\vec{i} + 2a\vec{j}}{2a\sqrt{t^2 + 1}} \\ &= \frac{t\vec{i} + \vec{j}}{\sqrt{t^2 + 1}} \end{aligned}$$

Now we have

$$\frac{dT}{dt} = \frac{i - tj}{(t^v + 1)^{3/2}}$$

$$\therefore \frac{dT}{ds} = \frac{\frac{dT}{dt}}{\frac{ds}{dt}} = \frac{i - tj}{(t^v + 1)^{3/2}} \cdot \frac{1}{2a(t^v + 1)^{1/2}}$$

$$\boxed{\frac{dT}{ds} = \frac{i - tj}{2a(t^v + 1)^2}}$$

W.K.T $\frac{dT}{ds} = K|N|$

$$\therefore \left| \frac{dT}{ds} \right| = K|N|$$

$$\Rightarrow K = \left| \frac{dT}{ds} \right| \quad (\because |N| = 1)$$

$$= \sqrt{\frac{1 + t^v}{4a^2(t^v + 1)^4}}$$

$$= \sqrt{\frac{1}{4a^2(t^v + 1)^3}} = \frac{1}{2a(t^v + 1)^{3/2}}$$

$$\therefore \frac{dT}{ds} = K N$$

$$\Rightarrow N = \frac{1}{K} \frac{dT}{ds} = \frac{2a (t^v+1)^{\frac{3}{2}} (i-tj)}{2a (t^v+1)^2}$$

$$= (t^v+1)^{-\frac{1}{2}} (i-tj)$$

$$N = \frac{i-tj}{(t^v+1)^{\frac{1}{2}}}$$

$$\therefore B = T \times N = \begin{vmatrix} i & j & k \\ t & -t & 0 \\ \frac{1}{\sqrt{t^v+1}} & \frac{1}{\sqrt{t^v+1}} & 0 \\ \frac{1}{\sqrt{t^v+1}} & \frac{-t}{\sqrt{t^v+1}} & 0 \end{vmatrix}$$

$$= k \left[\frac{-t^v}{t^v+1} - \frac{1}{t^v+1} \right]$$

$$= -k \left[\frac{t^v+1}{t^v+1} \right] = -k$$

$$\therefore B = -k$$

$$\therefore \frac{dB}{dt} = 0$$

$$\therefore \frac{dB}{ds} = -\tau N$$

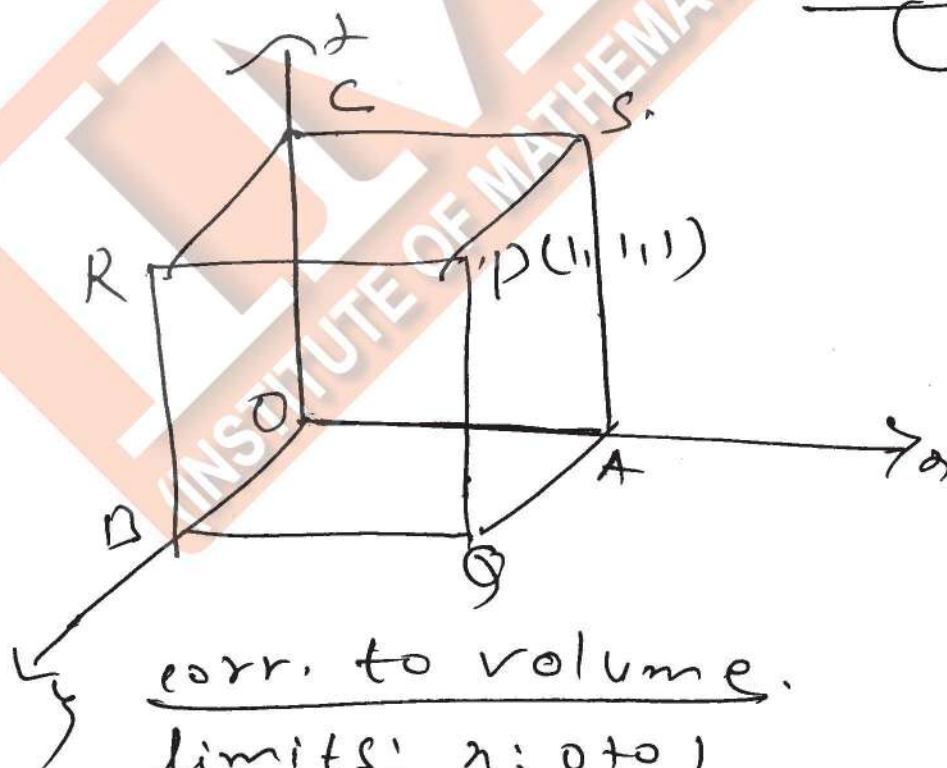
$$\therefore \frac{dB}{ds} = 0$$

$$\therefore \tau = \left| \frac{dB}{ds} \right| = 0$$

7(b) If $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$,
 then evaluate $\iint_S \vec{F} \cdot \vec{n} ds$,
 where 'S' the surface of a
 unit cube with two opposite
 corners at $(0,0,0)$ and $(1,1,1)$
 respectively.

Hence verify the divergence theorem

sol Gauss divergence theorem

$$\iint_S \vec{F} \cdot \vec{n} ds = \iiint_V \nabla \cdot \vec{F} dV.$$


limits: $x: 0 \text{ to } 1$
 $y: 0 \text{ to } 1$
 $z: 0 \text{ to } 1$

We have

$$\nabla \cdot \vec{F} = 4z - y.$$

$$\therefore \iiint_V \nabla \cdot \vec{F} \, dv = \int_{x=0}^1 \int_{y=0}^1 \left[\int_{z=0}^1 (4z - y) \, dz \right] dy \, dx$$

$$= \int_{x=0}^1 \int_{y=0}^1 \left[2z^2 - yz \right]_{z=0}^1 dy \, dx$$

$$= \int_{x=0}^1 \left[\int_{y=0}^1 (2 - y) \, dy \right] dx$$

$$= \int_{x=0}^1 \left(2y - \frac{y^2}{2} \right)_{y=0}^1 dx = \int_{x=0}^1 \left(2 - \frac{1}{2} \right) dx$$

$$= \int_{x=0}^1 \left(\frac{3}{2} \right) dx = \frac{3}{2} = 1.5$$

Let us find $\iint_S \vec{F} \cdot \hat{n} \, dS$:

Along $z=1$:

$$\text{Along } z=1: \quad \therefore \hat{n} = \hat{k}.$$

$$\therefore \vec{F} \cdot \hat{n} = yz = y \quad (\because z=1)$$

$$\text{and } dS = \frac{dx \, dy}{|\hat{n} \cdot \hat{k}|} = dx \, dy.$$

$$\therefore \iint_{S_1} \vec{F} \cdot \hat{n} \, dS = \left| \hat{n} \cdot \hat{k} \right| \int_{x=0}^1 \int_{y=0}^1 y \, dx \, dy = \frac{1}{2} \quad \text{--- (1)}$$

Along $z=0$

i.e. $S_2: z=0 \quad \therefore \hat{n} = -\hat{k}$.

$$\therefore \iint_{S_2} \vec{F} \cdot \hat{n} \, dS_2 = 0$$

Along $x=1$

i.e. $S_3: x-1=0$

$$\therefore \vec{F} \cdot \hat{n} = 4xz = 4z \quad (\because x=1)$$

$$\therefore dS_3 = \frac{dy \, dz}{|\hat{n} \cdot \hat{j}|} = dy \, dz$$

$$\therefore \iint_{S_3} \vec{F} \cdot \hat{n} \, dS_3 = \int_{y=0}^1 \left(\int_{z=0}^1 4z \, dz \right) dy$$

$$\boxed{\iint_{S_3} \vec{F} \cdot \hat{n} \, dS_3 = \int_{y=0}^1 (2) \, dy = 2}$$

Along $x=0$

$$\hat{n} = -\hat{i}$$

i.e. $S_4: x=0 \Rightarrow 0=0$

$$\therefore \iint_{S_4} \vec{F} \cdot \hat{n} \, dS_4 = 0$$

Along $y=1$

i.e. $S_5: y-1=0 \quad \therefore \hat{n} = \hat{j}$

$$dS_5 = \frac{dx \, dz}{|\hat{n} \cdot \hat{j}|} = dx \, dz$$

$$\therefore \vec{F} \cdot \hat{n} = -y^2 = -1 \quad (\because y=1)$$

$$\therefore \iint_{S_5} \vec{F} \cdot \hat{n} dS = \int_{x=0}^1 \int_{z=0}^1 (-1) dz dx$$

$$= -1.$$

Along $y=0$

i.e. $S_6: y=0 \quad \hat{n} = -\hat{j}$

$$\therefore \iint_{S_6} \vec{F} \cdot \hat{n} dS = 0.$$

$$\therefore \iint_{S_1+S_2+S_3+S_4+S_5+S_6} \vec{F} \cdot \hat{n} dS$$

$$= \iint_{S_1} \vec{F} \cdot \hat{n} dS + \iint_{S_2} \vec{F} \cdot \hat{n} dS$$

$$+ \iint_{S_3} \vec{F} \cdot \hat{n} dS + \iint_{S_4} \vec{F} \cdot \hat{n} dS$$

$$+ \iint_{S_5} \vec{F} \cdot \hat{n} dS + \iint_{S_6} \vec{F} \cdot \hat{n} dS$$

$$= \frac{1}{2} + 0 + 2 + 0 - 1 + 0$$

$$= \frac{1}{2} + 2 - 1 = \frac{3}{2} \text{ (LHS)}$$

$$\therefore \text{LHS} = \text{RHS}$$

$\therefore$ Gauss divergence theorem is verified

7(c) (i) S.O.T $(x+y+1)^{-4}$ is an integrating factor of $(2x-y-1)y dx + (2y-x-1)x dy \Rightarrow$.
 Hence solve it.

Sol Given that $(2x-y-1)y dx + (2y-x-1)x dy \Rightarrow$
 Let us show that $f(x,y) = (x+y+1)^{-4}$ is an integrating factor of (1).

$$\therefore \frac{(2x-y-1)y}{(x+y+1)^4} dx + x \frac{(2y-x-1)}{(x+y+1)^4} dy \Rightarrow$$

clearly it is in the form of $Mdx + Ndy \Rightarrow$ (2)

$$\therefore M = \frac{(2x-y-1)y}{(x+y+1)^4} ; N = \frac{x(2y-x-1)}{(x+y+1)^4}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{4(x+y+1)^3(2x-y-1)y - (x+y+1)^4(2x-2y-1)}{(x+y+1)^8}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{(x+y+1)^3 \left[4y(2x-y-1) - (x+y+1)(2x-2y-1) \right]}{(x+y+1)^8}$$

$$= \frac{4y(2x-y-1) - (x+y+1)(2x-2y-1)}{(x+y+1)^5}$$

we have

$$\frac{\partial N}{\partial x} = \frac{4(x+y+1)^3 \cdot (2y-x-1)x - (x+y+1)^4 (2y-2x-1)}{(x+y+1)^8}$$

$$= \frac{4x(2y-x-1) - (x+y+1)(2y-2x-1)}{(x+y+1)^5}$$

$$= 8xy - 4x^2 - 4x - \left[\begin{array}{l} 2xy - 2x^2 - x \\ + 2y^2 - 2xy - y \\ + 2y - 2x - 1 \end{array} \right]$$

$$= \frac{-2x^2 - 2y^2 + x + y + 2y - 2x - 1 + 8xy}{(x+y+1)^5}$$

$$= \frac{-2x^2 - 2y^2 - x + 3y + 8xy - 1}{(x+y+1)^5}$$

please you can

do some calculations of $\frac{\partial M}{\partial y}$ & $\frac{\partial N}{\partial x}$
 you will get $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

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$\therefore f(x+y)$

$f(x,y) = (x+y+1)^{-4}$ is an integrating factor.

$\therefore$ The g.s of is given by

$$\int \frac{(2x-y-1)y}{(x+y+1)^4} dx + \int 0 dy = C.$$

$$\Rightarrow \frac{xy}{(x+y+1)^3} = C.$$

which is required g.s
 of (i)

7(C)(ii) If $L\{f(t)\} = \bar{f}(s)$, then show that $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(u) du$.

Hence evaluate $L\left\{\frac{\cos at - \cos bt}{t}\right\}$.

Proof: Let $G(t) = \frac{1}{t} f(t)$

$$\Rightarrow f(t) = t G(t)$$

Apply Laplace Transform on both sides

$$L\{f(t)\} = L\{t G(t)\}$$

$$\Rightarrow \bar{f}(s) = -\frac{d}{ds} L\{G(t)\} \quad \left(\because L\{t F(t)\} = -f'(s)\right)$$

Integrating both sides w.r.t s from s to ∞ , we get

$$\int_s^\infty \bar{f}(s) ds = -\left[L\{G(t)\}\right]_s^\infty = -\left[\int_0^\infty e^{-st} G(t) dt\right]_s^\infty$$

$$\Rightarrow \int_s^\infty \bar{f}(s) ds = -\lim_{s \rightarrow \infty} \left[\int_0^\infty e^{-st} G(t) dt\right] +$$

$$\lim_{s \rightarrow s} \left[\int_0^\infty e^{-st} G(t) dt\right]$$

$$= 0 + \int_0^\infty e^{-st} G(t) dt$$

$$\left(\because \lim_{s \rightarrow \infty} \int_0^\infty e^{-st} G(t) dt = 0\right)$$

$$= L\{G(t)\}$$

$$\Rightarrow L\{G(t)\} = \int_s^\infty \bar{f}(s) ds = \int_s^\infty \bar{f}(u) du \quad \left(\because \text{by property of definite integrals}\right)$$

$$\therefore L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(u) du$$

Let $f(t) = \cos at - \cos bt$

$$\begin{aligned}\therefore \bar{f}(s) &= L\{f(t)\} = L\{\cos at - \cos bt\} \\ &= L\{\cos at\} - L\{\cos bt\} \\ &= \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\end{aligned}$$

By using the property

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(u) du$$

$$\begin{aligned}\therefore L\left\{\frac{\cos at - \cos bt}{t}\right\} &= \int_s^\infty \left(\frac{u}{u^2+a^2} - \frac{u}{u^2+b^2}\right) du \\ &= \frac{1}{2} \int_s^\infty \left(\frac{2u}{u^2+a^2} - \frac{2u}{u^2+b^2}\right) du \\ &= \frac{1}{2} \left[\log(u^2+a^2) - \log(u^2+b^2) \right]_s^\infty \\ &= \frac{1}{2} \left(\log\left(\frac{u^2+a^2}{u^2+b^2}\right) \right)_s^\infty \\ &= \frac{1}{2} \left[\lim_{u \rightarrow \infty} \log\left(\frac{1+\frac{a^2}{u^2}}{1+\frac{b^2}{u^2}}\right) - \log\left(\frac{s^2+a^2}{s^2+b^2}\right) \right] \\ &= \frac{1}{2} \left[\log 1 - \log \frac{s^2+a^2}{s^2+b^2} \right] \\ &= \frac{1}{2} \left[0 - \log \frac{s^2+a^2}{s^2+b^2} \right] \\ &= \frac{1}{2} \log \left(\frac{s^2+b^2}{s^2+a^2} \right) \\ \therefore L\left\{\frac{\cos at - \cos bt}{t}\right\} &= \frac{1}{2} \log \left(\frac{s^2+b^2}{s^2+a^2} \right)\end{aligned}$$

8 (a). Using the method of variation of parameters, solve the differential equation

$$\frac{d^2 y}{dx^2} + a^2 y = \sec ax.$$

SOLUTION

Given that,

$$\frac{d^2 y}{dx^2} + a^2 y = \sec ax \quad \dots(1)$$

$$(1) \equiv (D^2 + a^2)y = \sec ax$$

where $D = \frac{d}{dx}$

Auxiliary equation given by

$$m^2 + a^2 = 1 \Rightarrow m = \pm ia$$

$$\therefore \text{C. F} = C_1 \cos ax + C_2 \sin ax$$

Assuming $y_p = Au + Bv$

where $u = \cos ax; v = \sin ax$

u.v needs to be linearly independent

$$w(u,v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a \sin ax & a \cos ax \end{vmatrix} = a(\cos^2 ax + \sin^2 ax) = a(1)$$

$$a \neq 0$$

$\therefore$

$$A(x) = \int \frac{-vR}{w} dx$$

$$= \int \frac{-\sin ax \cdot \sec ax}{a} dx$$

$$A = \frac{-\log(\sec ax)}{a^2}$$

$$B = \int \frac{uR}{w} dx$$

$$= \int \frac{\cos ax \cdot \sec ax}{a} dx$$

$$= \frac{x}{a}$$

$\therefore$

$$y_p = \frac{-\log(\sec ax)}{a^2} \cdot \cos ax + \frac{x}{a} \sin ax$$

$\therefore$

$$y = y_c + y_p$$

$$y = c_1 \cos ax + c_2 \sin ax + \frac{x}{a} \sin ax$$

$$+ \frac{\log(\cos ax) \cos ax}{a^2}$$

Required Solution.

8(b) $\rightarrow$ AB is a uniform rod of length l and weight W , which can turn freely about a fixed point in its length distant $\frac{l}{3}$ from A. AC and BC are light strings each of length $\frac{5}{6}l$, attached to a particle C of weight w . Prove that if W is less than $2w$, there will be stable equilibrium with AB inclined to the horizontal at an angle $\tan^{-1}\left(\frac{W+w}{4w}\right)$.

Solⁿ: From the diagram

rod AB pivoted freely at fixed point O, located at $\frac{l}{3}$ from A. Two light strings AC and BC (each of length $\frac{5}{6}l$) run down to the particle C. weight W acts downward at the rod's midpoint, and weight w acts downward at C. The rods hits inclined at angle θ .

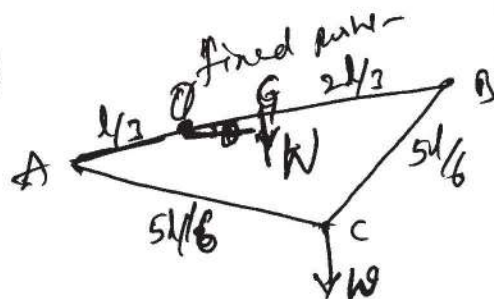
● = fixed pivot, $OA = \frac{l}{3}$

$OB = \frac{2}{3}l$. Since $AC = BC = \frac{5}{6}l$ and $AB = l$, the perpendicular distance from C to AB is

$$d = \sqrt{\left(\frac{5}{6}l\right)^2 - \left(\frac{l}{2}\right)^2} = \frac{2}{3}l$$

$$h_M = -\frac{l}{6} \sin \theta$$

$$h_C = -\frac{l}{6} \sin \theta - \frac{2}{3}l \cos \theta.$$



Total potential Energy

$$U(\theta) = Wh_M + wh_C = -\frac{l}{6}(W+w)\sin\theta - \frac{2l}{3}\cos\theta w$$

Equilibrium $\frac{dU}{d\theta} = 0$

$$-\frac{l}{6}(W+w)\cos\theta + \frac{2l}{3}w\sin\theta = 0$$

$$\Rightarrow \tan\theta = \frac{W+w}{4w} \Rightarrow \theta = \tan^{-1}\left(\frac{W+w}{4w}\right)$$

Stability:

$$\frac{d^2U}{d\theta^2} = -\frac{l}{6}(W+w)\sin\theta + \frac{2l}{3}w\cos\theta$$

Since θ is an acute angle in the first quadrant where both $\sin\theta > 0$ and $\cos\theta > 0$. and the physical constraints $W, w, l > 0$ hold true, the terms are strictly positive

$$\frac{d^2U}{d\theta^2} > 0.$$

Thus, the configuration represents a position of stable equilibrium.

8(c) (i) For a scalar point function ϕ and a vector point function $\vec{F}$, Prove that

$$\nabla \times (\phi \vec{F}) = (\nabla \phi) \times \vec{F} + \phi (\nabla \times \vec{F})$$

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$, then verify the above identity for $\phi = \frac{1}{r}$ and $\vec{F} = \vec{r}$.

Proof: we have

$$\text{curl}(\phi A) = \nabla \times (\phi A) = \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \times \phi A$$

$$= \sum \left\{ i \times \frac{\partial}{\partial x} (\phi A) \right\}$$

$$= \sum \left\{ i \times \left(\frac{\partial \phi}{\partial x} A + \phi \frac{\partial A}{\partial x} \right) \right\}$$

$$= \sum \left\{ i \times \left(\frac{\partial \phi}{\partial x} A \right) \right\} + \sum \left\{ i \times \left(\phi \frac{\partial A}{\partial x} \right) \right\}$$

$$= \sum \left\{ \left(\frac{\partial \phi}{\partial x} i \right) \times A \right\} + \sum \left\{ \phi \left(i \times \frac{\partial A}{\partial x} \right) \right\}$$

$$\left\{ \text{Note that } a \times (mb) = (ma) \times b = m(a \times b) \right\}$$

$$= \left\{ \sum \left(\frac{\partial \phi}{\partial x} i \right) \right\} \times A + \phi \sum \left(i \times \frac{\partial A}{\partial x} \right)$$

$$= (\nabla \phi) \times A + \phi (\nabla \times A)$$

Put $\phi = \frac{1}{r}$ and $\vec{F} = \vec{r}$

$$\text{Now } \nabla \times (\phi \vec{F}) = (\nabla \phi) \times \vec{F} + \phi (\nabla \times \vec{F})$$

replacing ϕ by $\frac{1}{r}$ and F by r in this identity, we have

$$\text{curl} \left[\left(\frac{1}{r} \right) r \right] = \left[\nabla \left(\frac{1}{r} \right) \right] \times r + \left(\frac{1}{r} \right) \text{curl } r$$

$$= \left[\left(-\frac{1}{r^2} \right) \nabla r \right] \times r + \left(\frac{1}{r} \right) 0$$

$$[\because \text{curl } r = 0 \text{ and } \nabla f(r) = f'(r) \nabla r]$$

$$[\because \nabla r = \frac{1}{r} r]$$

$$= \left[-\frac{1}{r^2} \frac{1}{r} r \right] \times r$$

$$= -\frac{1}{r^2} (r \times r)$$

$$= -\frac{1}{r^3} 0 = 0$$

f(xii) If $\vec{r} = a \cos t \hat{i} + a \sin t \hat{j} + b \hat{k}$
then show that

$$|\vec{r}' \times \vec{r}''|^2 = a^2 (a^2 + b^2)$$

$$\text{and } [\vec{r}' \cdot \vec{r}'' \cdot \vec{r}'''] = a^2 b.$$

Sol we have $\vec{r} = a \cos t \hat{i} + a \sin t \hat{j} + b \hat{k}$

$$\therefore \vec{r}' = -a \sin t \hat{i} + a \cos t \hat{j} + 0 \hat{k}$$

$$\vec{r}'' = -a \cos t \hat{i} - a \sin t \hat{j}$$

$$\vec{r}''' = a \sin t \hat{i} - a \cos t \hat{j}$$

We have $\vec{r}' \times \vec{r}'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a \sin t & a \cos t & b \\ -a \cos t & -a \sin t & 0 \end{vmatrix}$

$$= \hat{i}[-a \sin t] - \hat{j}[a \cos t] + \hat{k}[a^2]$$

$\therefore$ We have

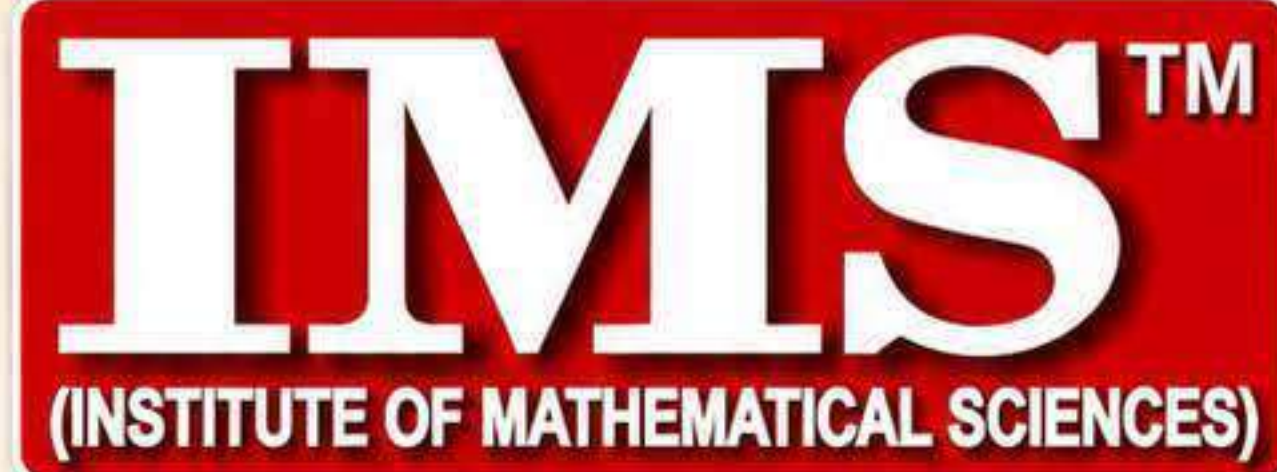
$$|\vec{r}' \times \vec{r}''| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + a^4}$$

$$\therefore |\vec{r}' \times \vec{r}''|^2 = a^2(a^2 + \sin^2 t + \cos^2 t)$$

We have

$$[\vec{r}' \quad \vec{r}'' \quad \vec{r}'''] = \begin{vmatrix} -a \sin t & a \cos t & b \\ -a \cos t & -a \sin t & 0 \\ a \sin t & -a \cos t & 0 \end{vmatrix}$$

$$= b(a^2) = b a^2$$



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