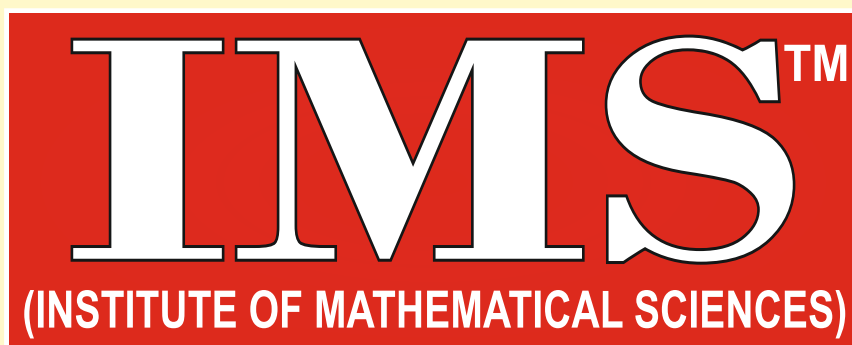




UPSC-CSE 2025

Mains

MATHEMATICS

Optional Paper-I

Solutions

1(a) Can the Set $\{(0,0,0,3), (1,1,0,0), (0,1,-1,0)\}$ be extended to form a basis of the vectorspace $\mathbb{R}^4$? Justify your answer.

Sol Let $\mathbb{R}^4 = \{(x,y,z,t) / x,y,z,t \in \mathbb{R}\}$ be a vector space over $\mathbb{R}$.

clearly $\dim(\mathbb{R}^4) = 4$.

Let $S = \{(0,0,0,3), (1,1,0,0), (0,1,-1,0)\}$ be a subset of $\mathbb{R}^4$.

we have

$$A = \begin{bmatrix} 0 & 0 & 0 & 3 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

clearly it is in the echelon form and has no zero rows.

$\therefore S$ is a L.I.

clearly S can be extended to form a basis of $\mathbb{R}^4$.

We know that

$$S' = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$$

is a standard basis of $\mathbb{R}^4$.

clearly $S'' = \{(1, 0, 0, 0), (0, 0, 0, 3), (1, 1, 0, 0), (0, 1, -1, 0)\} \subseteq \mathbb{R}^4$

we have

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

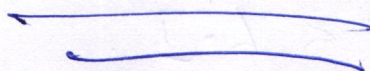
we have $|B| = \begin{vmatrix} 0 & 0 & 3 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{vmatrix}$

$$= 3(-1) = -3 \neq 0.$$

clearly S'' is L.I.

$\therefore$ Number of L.I. vectors = 4 =

$\therefore S''$ can be extended basis of $\mathbb{R}^4$ $\dim(\mathbb{R}^4)$



1.(b) Find the range, rank, kernel and nullity of the linear transformation $T: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ given by $T(x, y, z, w) = (x - w, y + z, z - w)$.

Sol Let $R(T) = \{ T(x, y, z, w) / x, y, z, w \in \mathbb{R} \}$
 $= \{ (x - w, y + z, z - w) / x, y, z, w \in \mathbb{R} \} \subseteq \mathbb{R}^3$

Let us find a subset $S \subseteq \mathbb{R}^3$
 s.t. $L(S) = \mathbb{R}^3$

Let $x = (x - w, y + z, z - w) \in R(T)$
 Then $x = x(1, 0, 0) + y(0, 1, 0) + z(0, 1, 1) + w(-1, 0, -1) \in L(S)$

where $S = \{ (1, 0, 0), (0, 1, 0), (0, 1, 1), (-1, 0, -1) \} \subseteq \mathbb{R}^3$.

we have

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$$

we have

$$A \cong \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$R_2 \rightarrow R_4 + R_2$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad R_3 \rightarrow R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_4 \rightarrow R_4 + R_3$$

clearly it is ~~is~~ clearly in the form of echelon form.

and has zero row, and three non-zero rows
 clearly $\therefore$ is L.O.D.

$$\therefore \text{Maximum number L.O.I} = 3 \\ = \dim(R(T)) \\ = \text{rank } T$$

$$\text{clearly } S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \in R(T).$$

$$\therefore R(T) = \{x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1) \mid x, y, z \in \mathbb{R}\} \\ = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}.$$

we have

$$N(T) = \left\{ (x, y, z, w) \mid T(x, y, z, w) = (0, 0, 0) \right\} \\ \text{in } \mathbb{R}^4$$

$$\text{Let } T(x, y, z, w) = (0, 0, 0)$$

$$\Rightarrow (x-w, y+z, z-w) = (0, 0, 0)$$

$$\Rightarrow \left. \begin{array}{l} x-w=0 \\ y+z=0 \\ z-w=0 \end{array} \right\} \Rightarrow \begin{array}{l} x=w \\ y=-z \\ z=w \end{array}$$

$$\Rightarrow \begin{array}{l} x=w \\ y=-w \\ z=w \end{array}$$

$$\therefore N(T) = \left\{ (w, -w, w, w) / w \in \mathbb{R} \right\}$$

is a nullspace of T .

Since there is only one free variable w $\therefore \dim(N(T)) = 1 = \text{nullity of } T$

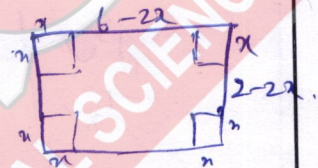
clearly basis of $N(T)$ is $\{(1, -1, 1, 1)\}$.

$$\text{is } \{(1, -1, 1, 1)\}.$$

100) A rectangular sheet of metal of length 6 meters and width 2 meters is given. Four equal squares are removed from the four corners. The sides of this sheet are now folded up to form an open rectangular box. Find approximately the height of the box, such that the volume of the box is maximum.

Solⁿ: Let x be the side length of the equal squares removed from each corner.

The original length of the sheet is 6 m, and the width is 2 m.



When the squares are removed and the sides are folded up, the dimensions of the open rectangular box are

$$l = 6 - 2x, \quad b = 2 - 2x, \quad h = x$$

$$\begin{aligned} \text{volume}(x) &= l b h \\ &= (6 - 2x)(2 - 2x)x \\ &= 4x^3 - 16x^2 + 12x \end{aligned}$$

$$\frac{dv}{dx} = \frac{d}{dx} (4x^3 - 16x^2 + 12x) = 12x^2 - 32x + 12$$

$$\text{for critical point, } \frac{dv}{dx} = 0$$

$$\Rightarrow 12x^2 - 32x + 12 = 0$$

$$\Rightarrow 3x^2 - 8x + 3 = 0$$

$$\Rightarrow x = \frac{4 \pm \sqrt{7}}{3}$$

$$x_1 = \frac{4 + \sqrt{7}}{3} \approx 2.215$$

$$x_2 = \frac{4 - \sqrt{7}}{3} \approx 0.451$$

Since x must be less than 1, the valid value for x is $\frac{4 - \sqrt{7}}{3}$.

$$\frac{dV}{dx} = 24x - 32$$

$$\left. \frac{dV}{dx} \right|_{x = \frac{4 - \sqrt{7}}{3}} = -8\sqrt{7} < 0.$$

$\therefore$ The volume is maximized at $x = \frac{4 - \sqrt{7}}{3}$.

1(d), Given that $f(x+y) = f(x)f(y)$ for all real x, y , $f(x) \neq 0$ for any real x and $f'(0) = 2$. Show that for all real x , $f'(x) = 2f(x)$. Hence find $f(x)$.

Sol'n: Given $f(x+y) = f(x)f(y)$; $f(0) \neq 0$ $\forall$

All real x, y ——— (i)

$f'(0) = 2$ ——— (ii)

Put $y = 0$,

$$f(x+0) = f(x) \cdot f(0)$$

$$\Rightarrow f(x) = f(x) \cdot f(0)$$

$$\Rightarrow f(0) = 1$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\therefore f'(x) = \lim_{h \rightarrow 0} \frac{f(x) \cdot f(h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} f(x) \left[\frac{f(h) - 1}{h} \right] = f(x)$$

$$= f(x) \lim_{h \rightarrow 0} \left[\frac{f(0+h) - f(0)}{h} \right]$$

$$= f(x) f'(0)$$

$$f'(x) = f(x) \cdot 2$$

$$\therefore f'(x) = 2f(x)$$

$$\text{Now } f'(x) = 2f(x)$$

$$\text{Now } \frac{1}{f(x)} \frac{d}{dx} f(x) = 2$$

$$\ln f(x) = 2x + C$$

$$\Rightarrow f(x) = e^{2x+C}$$

$$\Rightarrow f(x) e^C \cdot e^{2x}, f(0) = 1$$

put $x=0$,

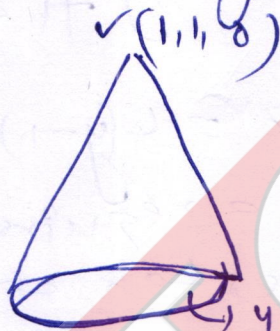
$$f(0) = e^C \cdot e^0$$

$$1 = e^C$$

$$f(x) = 1 \cdot e^{2x} = e^{2x}$$

1.(e) Find the equation of the cone whose vertex is the point $(1, 1, 0)$ and whose guiding curve is $y=0, x^2+z^2=4$.

Sol



Let $\frac{x-1}{l} = \frac{y-1}{m} = \frac{z-0}{n}$ be a generator.

If $\circ$ intersects the base curve $y=0, x^2+z^2=4$

$$\text{then } \frac{x-1}{l} = \frac{0-1}{m} = \frac{z-0}{n}$$

$$\Rightarrow \frac{x-1}{l} = \frac{-1}{m} = \frac{z}{n}$$

$$\Rightarrow x-1 = \frac{-l}{m} ; z = \frac{-n}{m}$$

$$\Rightarrow x = 1 - \frac{l}{m} ; z = \frac{-n}{m}$$

$$\therefore (x, y, z) = \left(1 - \frac{l}{m}, 0, \frac{-n}{m}\right)$$

Since it lies on $x^2+z^2=4$,

$$\therefore \left(1 - \frac{l}{m}\right)^2 + \left(\frac{n}{m}\right)^2 = 4$$

$$\Rightarrow \left(1 - \frac{x}{m}\right)^2 + \left(-\frac{y}{m}\right)^2 = 4, \Rightarrow (m-x)^2 + y^2 = 4m^2$$

clearly the locus of x, y, z given

by
$$[y-1 - (x-1)]^2 + (z-0)^2 = 4[y-1]^2$$

$$\Rightarrow (y-x)^2 + z^2 = 4(y-1)^2$$

which is a required cone
 with vertex at $(1, 1, 0)$

2.(a)

Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear transformation such that $T(1,1,-1) = (1,0)$
 $T(4,1,1) = (0,1)$ and $T(1,-1,2) = (1,1)$.
 Find T .

sol clearly $\dim(\mathbb{R}^3(\mathbb{R})) = 3$.

Let $S = \{(4,1,1), (1,-1,2), (1,1,-1)\} \subseteq \mathbb{R}^3$
 we have

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 4 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

we have

$$|A| = 1(2+1) - 1(8-1) - 1(-4-1) \\ = 3 - 7 + 5 = 1 \neq 0$$

clearly S is L.I

$\therefore$ Number of L.I vectors
 $= 3 = \dim(\mathbb{R}^3(\mathbb{R}))$.

$\therefore S$ is a basis of $\mathbb{R}^3$.

Let $(x,y,z) \in \mathbb{R}^3$ s.t

$$(x,y,z) = a(4,1,1) + b(1,-1,2) + c(1,1,-1) \\ \Rightarrow \begin{aligned} 4a + b + c &= x \\ a - b + c &= y \\ a + 2b - c &= z \end{aligned}$$

①

$$\Rightarrow \begin{bmatrix} 4 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow AX = B \text{ say}$$

where

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 4 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 5 & -3 \\ 0 & 3 & -2 \end{bmatrix} \quad \begin{array}{l} R_2 \rightarrow R_2 - 4R_1 \\ R_3 \rightarrow R_3 - R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 5 & -3 \\ 0 & 0 & -\frac{1}{5} \end{bmatrix} \quad \begin{array}{l} R_3 \rightarrow R_3 - \frac{3}{5}R_2 \\ -2 - \frac{3}{5}(-3) \\ -2 + \frac{9}{5} = -\frac{1}{5} \end{array}$$

$\therefore$ clearly it is in echelon form.

we have

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 5 & -3 \\ 0 & 0 & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow \begin{array}{l} a - b + c = x \\ 5b - 3c = y \\ -\frac{1}{5}c = z \Rightarrow c = -5z \end{array} \quad \begin{array}{l} 5b = 3(-5z) + y \Rightarrow 5b = -15z + y \\ b = -3z + \frac{y}{5} \end{array}$$

$$b = -3z + \frac{1}{5}y$$

$$a - (-3z + \frac{1}{5}y) - 5z = x$$

$$\Rightarrow a = x + (-3z + \frac{1}{5}y) + 5z$$

$$\Rightarrow \boxed{a = x + 2z + \frac{1}{5}y}$$

$$\therefore \textcircled{1} =$$

$$(x, y, z) = (x + \frac{1}{5}y + 2z)(4, 1, 1) + (-3z + \frac{1}{5}y)(1, -1, 2) + (-5z)(1, 1, -1)$$

Since $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is $L.T$

$$\therefore T(x, y, z) = (x + \frac{1}{5}y + 2z)T(4, 1, 1) + (-3z + \frac{1}{5}y)T(1, -1, 2) + (-5z)T(1, 1, -1)$$

$$\Rightarrow T(x, y, z) = (x + \frac{1}{5}y + 2z)(1, 0) + (-3z + \frac{1}{5}y)(0, 1) + (-5z)(1, 1)$$

$$\Rightarrow T(x, y, z) = \begin{pmatrix} x + \frac{1}{5}y + 2z - 5z, -3z + \frac{1}{5}y - 5z \end{pmatrix}$$

$$= \begin{pmatrix} x + \frac{1}{5}y - 3z, \frac{1}{5}y - 8z \end{pmatrix}$$

which is required T .

2(b) Using Mean value theorem, Prove that

$$\frac{\pi}{6} + \frac{\sqrt{3}}{15} < \sin^{-1}\left(\frac{3}{5}\right) < \frac{\pi}{6} + \frac{1}{8}$$

Solⁿ: Let $f(x) = \sin^{-1}x \forall x \in [a, b]$

where $a > 0; b < 1$

i.e. $0 < a < b < 1$

$f(x)$ is continuous & derivable in $[a, b]$ and

$$f'(x) = \frac{1}{\sqrt{1-x^2}} \forall x \in (a, b)$$

$\therefore$ By Lagrange's mean value theorem, $\exists c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow \frac{1}{\sqrt{1-c^2}} = \frac{\sin^{-1} b - \sin^{-1} a}{b - a} \quad \text{--- (1)}$$

Since $c \in (a, b)$

$$\Rightarrow a < c < b$$

$$\Rightarrow a^2 < c^2 < b^2$$

$$\Rightarrow -a^2 > -c^2 > -b^2$$

$$\Rightarrow 1 - a^2 > 1 - c^2 > 1 - b^2$$

$$\Rightarrow \sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2}$$

$$\Rightarrow \frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$$

$$\Rightarrow \frac{1}{\sqrt{1-a^2}} < \frac{\sin^{-1} b - \sin^{-1} a}{b-a} < \frac{1}{\sqrt{1-b^2}}$$

$$\Rightarrow \frac{b-a}{\sqrt{1-a^2}} < \sin^{-1} b - \sin^{-1} a < \frac{b-a}{\sqrt{1-b^2}} \quad \text{--- (I)}$$

putting $b = 3/5$, $a = 1/2$ then from (I)

$$\frac{3/5 - 1/2}{\sqrt{1 - 1/4}} < \sin^{-1} 3/5 - \sin^{-1} 1/2 < \frac{3/5 - 1/2}{\sqrt{1 - 9/25}}$$

$$\Rightarrow \frac{1}{10} \times \frac{2}{\sqrt{3}} < \sin^{-1}(3/5) - \pi/6 < \frac{1}{10} \times \frac{5}{4}$$

$$\Rightarrow \frac{1}{5\sqrt{3}} < \sin^{-1}(3/5) - \pi/6 < \frac{1}{8}$$

$$\Rightarrow \frac{\sqrt{3}}{5} + \frac{\pi}{6} < \sin^{-1}(3/5) < \frac{\pi}{6} + \frac{1}{8}$$

Q(11)

Find the equation of the cylinder whose generators are parallel to the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and that passes through the curve $x^2 + y^2 = 16, z = 0$.

Solⁿ: Let $P(x_1, y_1, z_1)$ be any point on the cylinder, then the equations of the generator through P are $\frac{x-x_1}{1} = \frac{y-y_1}{2} = \frac{z-z_1}{3}$

This generator meets the plane $z=0$ in the point given by

$$\frac{x-x_1}{1} = \frac{y-y_1}{2} = \frac{0-z_1}{3}$$

i.e. in the point $[x_1 - \frac{1}{3}z_1, y_1 - \frac{2}{3}z_1, 0]$

$\therefore$ The generator intersects the given curve if

$$(x_1 - \frac{1}{3}z_1)^2 + (y_1 - \frac{2}{3}z_1)^2 = 16$$

$\therefore$ the locus of $P(x_1, y_1, z_1)$ or the required equation of the cylinder is

$$(x - \frac{1}{3}z)^2 + (y - \frac{2}{3}z)^2 = 16$$

$$\Rightarrow x^2 + \frac{1}{9}z^2 - \frac{2}{3}xz + y^2 + \frac{4}{9}z^2 - \frac{4}{3}yz = 16$$

$$\Rightarrow 9x^2 + 9y^2 + z^2 - 12yz - 6zx - 144 = 0.$$

$$\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1} \text{ and } \frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$$

Then the direction ratios of the line p are
 $(2+3r) - (-2-3r')$, $8-r - (2r'-7)$, $3+r - (4r'+6)$
 $3r+3r'+6$, $-r-2r'+15$, $r-4r'-3$. ————— 9

and $-3(3r+3r'+6) + 2(-r-2r'+15) + 4(r-4r'-3) = 0$

$$\Rightarrow 11r + 7r' = 20 \quad \text{and} \quad 7r + 29r' = 0$$

And the direction ratios of the line PQ from (2) are $6, 15, -7$ or $2, 5, -1$.

(3) case b.i.v, i.e.,
now the required S.D = P.Q = $\sqrt{(3-1)^2 + (8-1)^2 + (3-6)^2}$
 $= \sqrt{16+25+9}$
 $= 7\sqrt{10}$

3(a)

Reduce the following matrix to echelon form:

$$A = \begin{bmatrix} 2 & -2 & 2 & 1 \\ -3 & 6 & 0 & -1 \\ 1 & -7 & 10 & 2 \end{bmatrix}$$

sol $A \sim \begin{bmatrix} 1 & -7 & 10 & 2 \\ -3 & 6 & 0 & -1 \\ 2 & -2 & 2 & 1 \end{bmatrix}$ $R_1 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & -7 & 10 & 2 \\ 0 & -15 & 30 & 5 \\ 0 & 12 & -18 & -3 \end{bmatrix}$$

$R_2 \rightarrow R_2 + 3R_1$
 $R_3 \rightarrow R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & -7 & 10 & 2 \\ 0 & 3 & 6 & 1 \\ 0 & -4 & -6 & -1 \end{bmatrix}$$

$R_2 \rightarrow \frac{1}{3} R_2$
 $R_3 \rightarrow \frac{4}{3} R_3$

$$\sim \begin{bmatrix} 1 & -7 & 10 & 2 \\ 0 & 3 & 6 & 1 \\ 0 & 0 & -14 & -\frac{7}{3} \end{bmatrix}$$

$R_3 \rightarrow R_3 - \frac{4}{3} R_2$
 $-6 - \frac{4}{3} \cdot 2$
 $-1 - 8$
 $-1 - \frac{4}{3} \cdot 1$

Clearly it is in the echelon form.

3.6)

Find the equations of the spheres which pass through the circle

$$x^2 + y^2 + z^2 - 2x + 2y + 4z - 3 = 0,$$

$2x + y + z = 4$ and touch the plane

$$3x + 4y = 14.$$

Sol Let the equation of the sphere through the circle

$$x^2 + y^2 + z^2 - 2x + 2y + 4z - 3 = 0,$$

$$2x + y + z = 4 \text{ be:}$$

$$x^2 + y^2 + z^2 - 2x + 2y + 4z - 3 + \lambda(2x + y + z - 4) = 0.$$

$$\Rightarrow x^2 + y^2 + z^2 + (2 + 2\lambda)x + (2 + \lambda)y + (4 + \lambda)z - 3 - 4\lambda = 0$$

R: Radius of (i) is

$$\sqrt{(1 + \lambda)^2 + \left(\frac{2 + \lambda}{2}\right)^2 + \left(\frac{4 + \lambda}{2}\right)^2 + 3 + 4\lambda}$$

C: Centre is:

$$\left(- (1 + \lambda), - \left(\frac{2 + \lambda}{2}\right), - \left(\frac{4 + \lambda}{2}\right)\right).$$



$$3x + 4y = 14$$

Since $3x+4y=14$ touches the plane sphere ①

∴ The distance from a centre to the plane ② equal to radius:

$$\frac{|3(-1+\lambda) + 4(-1-\frac{\lambda}{2}) - 14|}{\sqrt{9+16}} =$$

$$\sqrt{(1+\lambda)^2 + (\frac{2+\lambda}{2})^2 + (\frac{4+\lambda}{2})^2 + 3+4\lambda}$$

$$\Rightarrow \lambda = 0, \lambda = 2 \text{ (please do the calculations)}$$

$$\therefore \textcircled{1} \equiv x^2+y^2+z^2-2x+2y+4z-3=0$$

$$\text{and } x^2+y^2+z^2+(2)x+4y+6z-11=0$$

required spheres

3(C)i) Evaluate $\iint_R y \, dx \, dy$, where R is the region bounded by $y=x$ and $y=4x-x^2$.

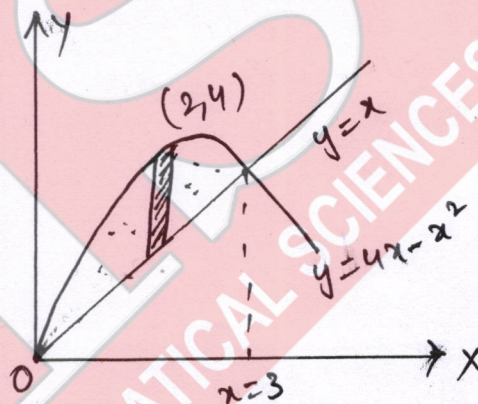
Sol'n: The equation of the parabola $y=4x-x^2$ may be written as $(x-2)^2 = -(y-4)$ i.e. this parabola has the vertex at the point $(2,4)$ and its concavity is downwards. The point of intersection of two curves are given as follows

$$4x - x^2 = x$$

$$\Rightarrow x^2 = 3x$$

$$\Rightarrow x(x-3) = 0$$

$$\Rightarrow x=0, x=3$$



and hence from $y=x$, we get-

$$y=0 \text{ at } x=0$$

$$y=3 \text{ at } x=3$$

$\therefore$ The points of intersection of the two curves are $(0,0)$, $(3,3)$.

The area can be considered as lying between the curves $y=x$, $y=4x-x^2$, $x=0$ and $x=3$.

So integrating along a vertical strip first i.e. y from $y=x$ to $y=4x-x^2$ and then w.r.t x from $x=0$ to $x=3$.

$$\begin{aligned}
 \therefore \text{The required area} &= \int_0^3 \int_{y=x}^{4x-x^2} y \, dy \, dx \\
 &= \int_0^3 \left[\frac{y^2}{2} \right]_x^{4x-x^2} dx \\
 &= \frac{1}{2} \int_0^3 \{ [4x-x^2]^2 - x^2 \} dx \\
 &= \frac{1}{2} \int_0^3 (x^4 - 8x^3 + 15x^2) dx \\
 &= \frac{1}{2} \left[\frac{x^5}{5} - \frac{8x^4}{4} + \frac{15x^3}{3} \right]_0^3 \\
 &= \frac{1}{2} \left[\frac{x^5}{5} - 2x^4 + 5x^3 \right]_0^3 \\
 &= \frac{1}{2} \left[\frac{243}{5} - 162 + 135 \right] \\
 &= \frac{1}{2} \left(\frac{243}{5} - 27 \right) \\
 &= \frac{1}{2} \left(\frac{108}{5} \right) \\
 &= \frac{54}{5} \\
 &= 10.8
 \end{aligned}$$

3(C)ii)

If $u(x, y) = xf\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$, where f and g are arbitrary functions then show that-

$$I. \quad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = xf\left(\frac{y}{x}\right)$$

$$II. \quad x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0.$$

Sol'n: I. Let $u_1 = xf\left(\frac{y}{x}\right)$, $u_2 = g\left(\frac{y}{x}\right)$. Then $u = u_1 + u_2$, where u_1 and u_2 are homogeneous functions of x and y of degrees 1 and 0 respectively. By Euler's theorem.

$$x \frac{\partial u_1}{\partial x} + y \frac{\partial u_1}{\partial y} = u_1 \quad \text{--- (1)}$$

$$x \frac{\partial u_2}{\partial x} + y \frac{\partial u_2}{\partial y} = 0, \quad u_2 = 0 \quad \text{--- (2)}$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = x \frac{\partial}{\partial x} (u_1 + u_2) + y \frac{\partial}{\partial y} (u_1 + u_2)$$

$$= x \frac{\partial u_1}{\partial x} + x \frac{\partial u_2}{\partial x} + y \frac{\partial u_1}{\partial y} + y \frac{\partial u_2}{\partial y}$$

$$= u_1 + 0, \text{ using (1) \& (2)}$$

$$\therefore x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = u_1 \quad \text{--- (3)}$$

Differentiating (3) partially w.r.t x and y respectively, we obtain

$$\frac{\partial u}{\partial x} + x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u_1}{\partial x} \quad \text{--- (4)}$$

$$x \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} + y \frac{\partial^2 u}{\partial y^2} = \frac{\partial u_1}{\partial y} \quad \text{--- (5)}$$

Multiplying (4) by x , (5) by y and adding, we obtain

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} \\ = x \frac{\partial u_1}{\partial x} + y \frac{\partial u_1}{\partial y} = u_1, \text{ by (1)}$$

Hence $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0$ using (2).

4(B) Show that there is no tangent plane to the sphere $x^2 + y^2 + z^2 - 4x + 2y - 4z + 4 = 0$ that can be passed through the straight line

$$\frac{x+6}{2} = \frac{y+3}{1} = \frac{z+1}{1}$$

Sol



Given that $x^2 + y^2 + z^2 - 4x + 2y - 4z + 4 = 0$

$$\Rightarrow (x-2)^2 + (y+1)^2 + (z-2)^2 = 1 \quad (1)$$

Centre C: (2, -1, 2)

Radius R = 1.

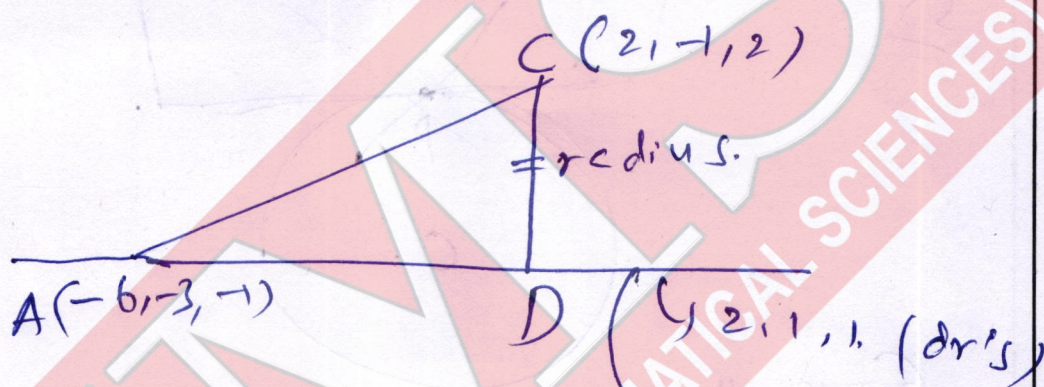
Given that

$$\frac{x+6}{2} = \frac{y+3}{1} = \frac{z+1}{1} \quad (2)$$

clearly the line is passing through the point: (-6, -3, -1)

and the dir's of a line 2, 1, 1.

We know that for a tangent plane to the sphere to pass through the given straight line, the distance from the centre of the sphere to the line must be equal to the radius of the sphere.



$$AD = \left| \frac{2}{\sqrt{6}}(2+6) + \frac{1}{\sqrt{6}}(2) + \frac{1}{\sqrt{6}}(2+1) \right|$$

$$= \left| \frac{16}{\sqrt{6}} + \frac{2}{\sqrt{6}} + \frac{3}{\sqrt{6}} \right| = \frac{21}{\sqrt{6}}$$

$$\therefore AC = \sqrt{64 + 4 + 9}$$

$$= \sqrt{77}$$

$$\therefore (CD)^2 = (AC)^2 - (AD)^2 = 77 - \frac{(21)^2}{6}$$

$$= \frac{462 - 441}{6}$$

$$= \frac{21}{6} = \frac{7}{2}$$

$$CD = \sqrt{7/2}.$$

$\neq$ radius. ($r=1$)

$\therefore$ No tangent plane to
the sphere can be passed through
the given line.

4(b) If $f(x, y) = \begin{cases} xy \frac{x^2 - y^2}{x^2 + y^2}, & \text{when } (x, y) \neq (0, 0) \\ 0, & \text{when } (x, y) = (0, 0) \end{cases}$

then find $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$.

Sol'n: we have

$$f_{xy}(0, 0) = \lim_{h \rightarrow 0} \frac{f_y(0+h, 0) - f_y(0, 0)}{h}$$

$$\text{Now } f_y(0, 0) = \lim_{k \rightarrow 0} \frac{f(0, 0+k) - f(0, 0)}{k} = \lim_{k \rightarrow 0} \frac{0}{k} = 0$$

$$\text{and } f_y(h, 0) = \lim_{k \rightarrow 0} \frac{f(h, 0+k) - f(h, 0)}{k}$$

$$= \lim_{k \rightarrow 0} \frac{hk(h^2 - k^2)}{k(h^2 + k^2)} = h$$

$$\therefore f_{xy}(0, 0) = \lim_{h \rightarrow 0} \frac{h-0}{h} = 1$$

$$\text{Again } f_{yx}(0, 0) = \lim_{k \rightarrow 0} \frac{f_x(0, 0+k) - f_x(0, 0)}{k}$$

$$\text{Now } f_x(0, 0) = \lim_{h \rightarrow 0} \frac{f(0+h, 0) - f(0, 0)}{h} = 0$$

$$\text{and } f_x(0, k) = \lim_{h \rightarrow 0} \frac{f(0+h, k) - f(0, k)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{hk(h^2 - k^2)}{h(h^2 + k^2)} = -k$$

$$\therefore f_{yx}(0, 0) = \lim_{k \rightarrow 0} \frac{-k-0}{k} = -1$$

$$\text{thus } \underline{\underline{f_{xy}(0, 0) \neq f_{yx}(0, 0)}}$$

20(i)
7

~~Let~~ Find the eigen values and the corresponding eigen values of the matrix:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -6 \\ 2 & -2 & 3 \end{bmatrix}$$

sol The characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & 1-\lambda & -6 \\ 2 & -2 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda) [(1-\lambda)(3-\lambda) - 12] - 2[6 - 2\lambda + 12] = 0$$

$$\Rightarrow (1-\lambda) [3 + \lambda^2 - 4\lambda - 12] - 2[-2\lambda + 18] = 0$$

$$\Rightarrow (1-\lambda) [\lambda^2 - 4\lambda - 9] + 4\lambda - 36 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda - 9 - \lambda^3 + 4\lambda^2 + 9\lambda + 4\lambda - 36 = 0$$

$$\Rightarrow -\lambda^3 + 5\lambda^2 + 9\lambda - 45 = 0$$

$$\Rightarrow \lambda = -3, 3, 5$$

are required characteristic roots.

Let us find characteristic vectors corresponding vectors:

If $\lambda = -3$:

Then we have

$$(A - \lambda I)X = 0$$

$$\Rightarrow \begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & -6 \\ 2 & -2 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 4 & -6 \\ 4 & 2 & 0 \\ 2 & -2 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_2 \leftrightarrow R_1$$

$$\Rightarrow \begin{bmatrix} 2 & 4 & -6 \\ 0 & -6 & 12 \\ 0 & -6 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - R_2 \end{array}$$

$$\Rightarrow \begin{bmatrix} 2 & 4 & -6 \\ 0 & -6 & 12 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3 \rightarrow R_3 - R_2$$

$$\Rightarrow 2x + 4y - 6z = 0 \Rightarrow x = -2z$$

$$-6y + 12z = 0 \Rightarrow y = 2z$$

If $z = k$ then $x = -2k$, $y = 2k$

$$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2k \\ 2k \\ k \end{bmatrix} = k \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} = kX_1$$

where $X_1 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$ required characteristic vector.
 sly you can try other two.

4 CQI
 →

Let P_n denote the vector space of all polynomials of degree $\leq n$ over $\mathbb{R}$.

verify that

$$\dim\left(\frac{P_4}{P_2}\right) = \dim(P_4) - \dim(P_2)$$

Solⁿ: we have

$$P_2 = \{a_0 + a_1x + a_2x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\} \text{ and}$$

$$P_4 = \{a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \mid a_0, a_1, a_2, a_3, a_4 \in \mathbb{R}\}$$

It is clear that P_2 is subspace of P_4 .

WKT $\{1, x, x^2\}$ & $\{1, x, x^2, x^3, x^4\}$ are bases of P_2 and P_4 respectively.

$$\therefore \dim(P_2) = 3 \quad ; \quad \dim(P_4) = 5$$

Now we define

$$\begin{aligned} \frac{P_4}{P_2} &= \{P_2 + a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \mid a_i \in \mathbb{R}\} \\ &= \{(a_3x^3 + a_4x^4) + P_2 + (a_0 + a_1x + a_2x^2) \mid a_i \in \mathbb{R}\} \\ &= \{a_3x^3 + a_4x^4 + P_2 \mid a_3, a_4 \in \mathbb{R}\} \end{aligned}$$

$$\begin{aligned} &(\because \alpha \in W \Rightarrow W + \alpha = W \\ &\text{i.e., } \alpha = a_0 + a_1x + a_2x^2 \in P_2 \\ &\alpha \in P_2 \Rightarrow P_2 + \alpha \in P_2) \end{aligned}$$

$\therefore \frac{P_4}{P_2}$ is a quotient space.

P_2 with the basis $\{P_2 + x^3, P_2 + x^4\}$

$$\therefore \dim\left(\frac{P_4}{P_2}\right) = 2$$

$$= \dim(P_4) - \dim(P_2)$$

5(a) solve $\left(1 - y^2 + \frac{y^4}{x^2}\right) \left(\frac{dy}{dx}\right)^2 - 2 \frac{y}{x} \frac{dy}{dx} + \frac{y^2}{x^2} = 0$

Sol'n: Rewriting the given equation, we have

$$p^2 - \left(\frac{2y}{x}\right)p + \left(\frac{y}{x}\right)^2 = p^2 y^2 - \left(\frac{p^2 y^4}{x^2}\right)$$

$$\Rightarrow (p - y/x)^2 = p^2 y^2 (1 - y^2/x^2)$$

$$px - y = \pm py (x^2 - y^2)^{1/2}$$

$$\{x \mp y(x^2 - y^2)^{1/2}\} \frac{dy}{dx} = y \quad (\text{or}) \quad \frac{dx}{dy} = \frac{x \mp (x^2 - y^2)^{1/2}}{y} \quad \text{--- (1)}$$

To solve it, put $x = vy$ so that $\frac{dx}{dv} = v + y \left(\frac{dv}{dy}\right)$

Then (1) gives

$$v + y \frac{dv}{dy} = \frac{vy \mp \sqrt{(v^2 y^2 - y^2)}}{y} = v \mp \sqrt{v^2 - 1}$$

$$\frac{dv}{dy} = \mp \sqrt{v^2 - 1}$$

$$\Rightarrow \frac{dv}{\sqrt{v^2 - 1}} = \mp dy$$

Integrating $\cosh^{-1} v = C \mp y$

$$\Rightarrow \cosh^{-1}(x/y) = C \mp y$$

5(b)

Form the differential equation of all ellipses whose axes coincide with coordinate axes.

Solⁿ: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ — (1)

Here a & b are arbitrary constants.
 we have to eliminate the constants a & b

$\Rightarrow b^2 x^2 + y^2 a^2 = a^2 b^2$ — (2)

differentiate (2) w.r.t x

$2b^2 x + 2y y' a^2 = 0$

$\Rightarrow b^2 x + a^2 y y' = 0$ — (3)

differentiating (3) w.r.t x

$b^2 + a^2 y y'' + a^2 y'^2 = 0$

$\Rightarrow b^2 = -a^2 (y y'' + y'^2)$ — (4)

from (3) & (4)

$-a^2 (y y'' + y'^2) x + x^2 y y' = 0$

$\Rightarrow -a^2 (x y y'' + x y'^2 - y y') = 0$

$\Rightarrow x y y' + x y'^2 - y y' = 0$

which is the required differential equation.

5(C)

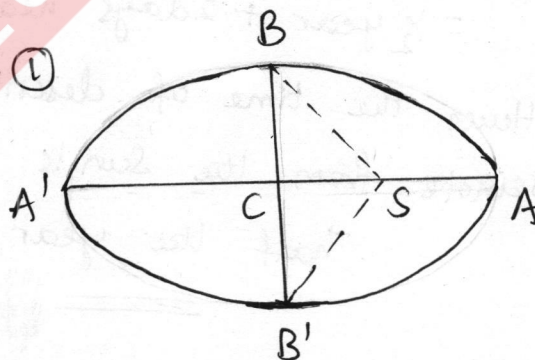
Prove that the time taken by the Earth to travel over half of its orbit, which is separated by the minor axis and is remote from the Sun, when the Sun is at the focus of the elliptic point, is twodays more than half of the year. The eccentricity of the orbit is taken as $\frac{1}{60}$.

Solⁿ: In the figure the path of the earth round the Sun is an ellipse, the Sun being at the focus S. BB' is the minor axis of the ellipse and C the centre of the ellipse. Remote half from the Sun separated by the minor axis is the arc $BA'B'$.

We know that in a central orbit the rate of description of the sectorial area is constant and is equal to $\frac{h}{2}$. The whole area of the ellipse is πab and the earth takes one year time to describe the elliptic orbit round the Sun.

$$\therefore \text{one year} = \frac{\pi ab}{\frac{h}{2}} \quad \text{--- (1)}$$

The sectorial area traced out by the earth while describing the area $BA'B'$.



$$= \text{the area } SBA'B'S$$

$$= \frac{1}{2} \text{ area of the ellipse} + \text{area of the } \triangle SBB'$$

$$= \frac{1}{2} \pi ab + \frac{1}{2} BB' \cdot CS$$

$$= \frac{1}{2} \pi ab + \frac{1}{2} \cdot 2b \cdot ae = \frac{1}{2} \pi ab + abe.$$

If t be the time taken by the earth to describe the arc $BA'B'$, then

$$t = \frac{\text{the Sectorial area } SBA'B'S}{h/2}$$

$$\text{i.e., } t = \frac{\frac{1}{2} \pi ab + abe}{h/2} \quad \text{--- (2)}$$

Dividing (2) by (1) we have

$$\frac{t}{\text{one year}} = \frac{\frac{1}{2} \pi ab + abe}{\pi ab} = \frac{1}{2} + \frac{e}{\pi}$$

$$\therefore t = \left(\frac{1}{2} + \frac{e}{\pi} \right) \times \text{one year} = \frac{1}{2} \text{ year} + \frac{e}{\pi} \text{ year}$$

$$= \frac{1}{2} \text{ year} + \frac{1}{60\pi} \text{ year} \quad [\because e = \frac{1}{60}]$$

$$= \frac{1}{2} \text{ year} + \left[\frac{1}{60} \times \frac{7}{22} \times 365 \right] \text{ days} \quad [\because \text{one year} = 365 \text{ days}]$$

$$= \frac{1}{2} \text{ year} + 2 \text{ days nearly.}$$

Hence the time of describing half of the orbit-
 remote from the Sun is two days more than
 half the year.

5(d) A and B are two points in the same horizontal line distant $2a$ apart. AO, OB are two equal heavy strings tied together at O and carrying a weight at O. If l is the length of each string, d the depth of O below AB, show that the parameter c of the catenary in which either string hangs is given by

$$l^2 - d^2 = 2c^2 [\cosh(a/c) - 1].$$

Solⁿ: Let C be the vertex of the catenary of which AO is an arc and c be its parameter. Let $O'x$ be the directrix and $O'y$ the axis of this catenary. Referred to $O'x$ and $O'y$ as the coordinates axes, let (x_1, y_1) and (x_2, y_2) be the coordinates of the points O and A respectively and let arc $CO = b$. Then

$$\begin{aligned} \text{arc } CA &= \text{arc } CO + \text{arc } OA \\ &= b + l \end{aligned}$$

Given that

$$OD = d \text{ and } AB = 2a$$

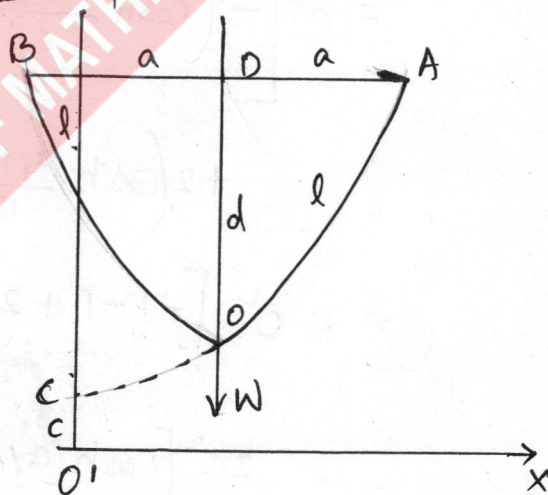
$$\text{So that } AD = a$$

$$\text{we have } y_2 = y_1 + d$$

$$\text{and } x_2 = x_1 + DA = x_1 + a$$

$$b = c \sinh \frac{x_1}{c}, \text{ [for the point O]}$$

$$\text{and } b + l = c \sinh \frac{x_2}{c} \text{ [for the point A]}$$



Subtracting, we have

$$l = c \left[\sinh \frac{x_1 + a}{c} - \sinh \frac{x_1}{c} \right] \quad [\because x_2 = x_1 + a]$$

Also from $y = c \cosh \frac{x}{c}$, we have

$$y_1 = c \cosh \frac{x_1}{c} \text{ and } y_2 = c \cosh \frac{x_2}{c}$$

Subtracting, we have

$$y_2 - y_1 = c \left[\cosh \frac{x_2}{c} - \cosh \frac{x_1}{c} \right]$$

$$(or) d = c \left[\cosh \frac{x_1 + a}{c} - \cosh \frac{x_1}{c} \right]$$

$$\begin{aligned} \therefore l^2 - d^2 &= c^2 \left[\left(\sinh \frac{x_1 + a}{c} - \sinh \frac{x_1}{c} \right)^2 - \left(\cosh \frac{x_1 + a}{c} - \cosh \frac{x_1}{c} \right)^2 \right] \\ &= c^2 \left[- \left(\cosh^2 \frac{x_1 + a}{c} - \sinh^2 \frac{x_1 + a}{c} \right) - \left(\cosh^2 \frac{x_1}{c} - \sinh^2 \frac{x_1}{c} \right) \right. \\ &\quad \left. + 2 \left(\cosh \frac{x_1 + a}{c} \cdot \cosh \frac{x_1}{c} - \sinh \frac{x_1 + a}{c} \cdot \sinh \frac{x_1}{c} \right) \right] \\ &= c^2 \left[-1 - 1 + 2 \cosh \left(\frac{x_1 + a}{c} - \frac{x_1}{c} \right) \right] \\ &= 2c^2 [\cosh(a/c) - 1] \end{aligned}$$

=====

5(c) → If $u = x + y + z$, $v = x^2 + y^2 + z^2$ and $w = xy + yz + zx$ then show that $\text{grad } u$, $\text{grad } v$ and $\text{grad } w$ are coplanar.

Solⁿ: Given $u = x + y + z$, $v = x^2 + y^2 + z^2$,
 $w = xy + yz + zx$.

$$\text{grad } u = \nabla u = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (x + y + z)$$

$$= \hat{i} \frac{\partial}{\partial x} (x + y + z) + \hat{j} \frac{\partial}{\partial y} (x + y + z) + \hat{k} \frac{\partial}{\partial z} (x + y + z)$$

$$\nabla u = \hat{i} + \hat{j} + \hat{k}$$

$$\text{Now grad } v = \hat{i} \frac{\partial}{\partial x} (x^2 + y^2 + z^2) + \hat{j} \frac{\partial}{\partial y} (x^2 + y^2 + z^2) + \hat{k} \frac{\partial}{\partial z} (x^2 + y^2 + z^2)$$

$$\nabla v = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\text{Now grad } w = \hat{i} \frac{\partial}{\partial x} (yz + zx + xy) + \hat{j} \frac{\partial}{\partial y} (yz + zx + xy) + \hat{k} \frac{\partial}{\partial z} (yz + zx + xy)$$

$$\nabla w = (y + z)\hat{i} + (z + x)\hat{j} + (x + y)\hat{k}$$

To prove that ∇u , ∇v and ∇w coplanar, we must have the following condition to be true.

$$\text{i.e. } \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix} = 0$$

On carrying out operations on LHS, we get

$$C_1 \rightarrow C_1 - C_2$$

$$C_2 \rightarrow C_2 - C_3, \text{ we get}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ 2(x-y) & 2(y-z) & 2z \\ y-x & z-y & x+y \end{vmatrix}$$

Solving the determinant we get

$$= 1 [2(x-y)(z-y) - 2(y-z)(y-x)]$$

$$= 2 [(x-y)(z-y) - (x-y)(z-y)]$$

$$= 0 = \text{RHS}$$

Hence, we can say that

grad u, grad v, and grad w are coplanar

6(a) If $F(s)$ and $G(s)$ are Laplace transforms of $f(t)$ and $g(t)$ respectively, then prove that-

$$L \left\{ \int_0^t f(x) g(t-x) dx \right\} = F(s) G(s).$$

Using this result, solve the equation

$$y(t) = t + \int_0^t y(x) \sin(t-x) dx.$$

Sol: Given that-

$$F(s) = L\{f(t)\} \quad \text{and} \quad G(s) = L\{g(t)\}$$

we have to show that-

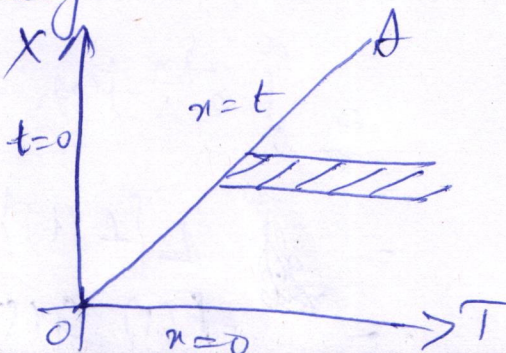
$$L \left\{ \int_0^t f(x) g(t-x) dx \right\} = F(s) G(s).$$

$$\text{Let } H(t) = \int_0^t f(x) g(t-x) dx$$

$$\text{Since } L\{H(t)\} = \int_0^\infty e^{-st} H(t) dt$$

$$\therefore L\{H(t)\} = \int_0^\infty e^{-st} \left\{ \int_0^t f(x) g(t-x) dx \right\} dt \quad \text{by defn of LT} \quad \text{--- (1)}$$

In equation (1), the region of integration in the double integral is the infinite area below the line OA (with equation $x=t$) and above the line OT



(with equation $x=0$). Here t & x are measured along OT and OX respectively.

To Change the order of Integration.

Draw an elementary strip parallel to T -axis, one end lies on $x=t$ and other end on $t=\infty$.

For the strip x varies from 0 to ∞
Hence changing the order of integration

① reduces to

$$L\{f(t)\} = \int_{x=0}^{\infty} f(x) \left\{ \int_{t=x}^{\infty} e^{-pt} g(t-x) dt \right\} dx$$

$$= \int_{x=0}^{\infty} f(x) \left\{ \int_{y=0}^{\infty} e^{-p(x+y)} g(y) dy \right\} dx$$

by putting $t-x=y$
 $t = y+x$
 $dt = dy$

$$= \int_{x=0}^{\infty} f(x) \left\{ e^{-px} \int_{y=0}^{\infty} e^{-py} g(y) dy \right\} dx$$

limits of y : 0 to ∞

$$= \int_{x=0}^{\infty} e^{-px} f(x) dx \left[\int_{y=0}^{\infty} e^{-py} g(y) dy \right]$$

$$= L\{f(t)\} L\{g(t)\}$$

$$= F(s) G(s)$$

Given $y(t) = t + \int_0^t y(x) \sin(t-x) dx$ — (1)

Using the defn of convolution, (1) can be written as

$$y(t) = t + y(t) * \sin t$$

Taking the LT of both the sides, we have

$$\begin{aligned} L\{y(t)\} &= L\{t\} + L\{y(t) * \sin t\} \\ &= \frac{1}{p^2} + L\{y(t)\} \times \frac{1}{p^2+1} \end{aligned}$$

$$\left(1 - \frac{1}{p^2+1}\right) L\{y(t)\} = \frac{1}{p^2}$$

$$\frac{p^2}{p^2+1} L\{y(t)\} = \frac{1}{p^2}$$

$$L\{y(t)\} = \frac{1}{p^2} \cdot \frac{p^2+1}{p^2} = \frac{p^2+1}{p^4}$$

$$L\{y(t)\} = \frac{1}{p^2} + \frac{1}{p^4}$$

$$y(t) = L^{-1}\left\{\frac{1}{p^2}\right\} + L^{-1}\left\{\frac{1}{p^4}\right\}$$

$$= t + \frac{t^3}{3!}$$

$$= t + \frac{t^3}{6}$$

6.(b) A heavy particle is attached to one end of an elastic string, the other end of which is fixed. The modulus of elasticity of the string is equal to the weight of the particle. The string is drawn vertically down till it is four times its natural length and then let go. Show that the particle will return to this point in time $\sqrt{\frac{a}{g} \left[\frac{4\pi}{3} + 2\sqrt{3} \right]}$, where a is the natural length of the string.

Sol'n: Let $OA = a$ be the natural length of an elastic string whose one end is fixed at O . Let B be the position of equilibrium of a particle of mass m attached to the other end of the string and $AB = d$. If T_B is the tension in the string OB , then by

$$\text{Hooke's law, } T_B = \lambda \frac{OB - OA}{OA} = \lambda \frac{d}{a}$$

where λ is the modulus of elasticity of the string. Considering the equilibrium of the particle at B , we have

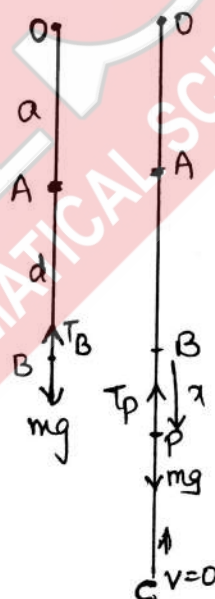
$$mg = T_B = \lambda \frac{d}{a} = mg \frac{d}{a} \quad [\because \lambda = mg, \text{ as given}]$$

$$\therefore d = a$$

Now the particle is pulled down to a point C such that $OC = 4a$ and then let go. It starts moving towards B with velocity zero at C . Let P be the position of the particle at time t , where $BP = x$

when the particle is at P , there are two forces acting upon it.

- (i) The tension $T_P = \lambda \frac{a+x}{a} = \frac{mg}{a} (a+x)$ in the string OP acting in the direction PO , i.e. in the direction of x decreasing
- (ii) the weight mg of the particle acting vertically downwards i.e. in the direction of x increasing.



Hence by Newton's second law of motion ($P=mf$), the equation of motion of the particle at P is

$$m \frac{d^2x}{dt^2} = mg - \frac{mg}{a} (a+x) = -\frac{mgx}{a}$$

$$\text{Thus } \frac{d^2x}{dt^2} = -\frac{g}{a}x \quad \text{--- (1)}$$

which is the equation of S.H.M with centre at the origin B and the amplitude $BC=2a$ which is greater than $AB=a$. Multiplying both sides of (1) by $2(dx/dt)$ and integrating w.r.t 't', we have

$$\left(\frac{dx}{dt}\right)^2 = -\frac{g}{a}x^2 + k, \quad \text{where } k \text{ is a constant.}$$

At the point C, $x = BC = 2a$, and the velocity $dx/dt = 0$

$$\therefore k = \frac{g}{a}4a^2$$

$$\therefore \left(\frac{dx}{dt}\right)^2 = \frac{g}{a}(4a^2 - x^2) \quad \text{--- (2)}$$

Taking square root of (2), we have

$$dx/dt = -\sqrt{\left(\frac{g}{a}\right)} \sqrt{4a^2 - x^2}$$

the -ve sign has been taken because the particle is moving in the direction of x decreasing.

Separating the variables, we have

$$dt = -\sqrt{\frac{a}{g}} \frac{dx}{\sqrt{4a^2 - x^2}} \quad \text{--- (3)}$$

If t_1 be the time from C to A, then integrating (3) from C to A, we get-

$$\int_0^{t_1} dt = -\sqrt{\left(\frac{a}{g}\right)} \int_{2a}^a \frac{dx}{\sqrt{4a^2 - x^2}}$$

$$\Rightarrow t_1 = \sqrt{\frac{a}{g}} \left[\cos^{-1} \frac{x}{2a} \right]_{2a}^a = \sqrt{\frac{a}{g}} \left[\cos^{-1} \left(-\frac{1}{2}\right) - \cos^{-1}(1) \right] = \sqrt{\frac{a}{g}} \cdot \frac{2\pi}{3}$$

Let v_1 be the velocity of the particle at A, then at A
 $x = -a$ and $(dx/dt)^2 = v_1^2$
 So from (2), we have $v_1^2 = (g/a)(4a^2 - a^2)$
 $\Rightarrow v_1 = \sqrt{3ag}$, the direction of v_1 being
 vertically upwards. Thus the velocity at A is $\sqrt{3ag}$ and is in
 the upwards direction so that the particle rises above A.
 Since the tension of the string vanishes at A, therefore
 at A the simple harmonic motion ceases and the particle
 when rising above A moves freely under gravity. Thus the
 particle rising from A with velocity $\sqrt{3ag}$ moves upwards
 till this velocity is destroyed. The time t_2 for this motion
 is given by $0 = \sqrt{3ag} - gt_2$. So that $t_2 = \sqrt{\frac{3a}{g}}$
 Conditions being the same, the equal time t_2 is taken by
 the particle in falling freely back to A. From A to C the
 particle will take the same time t , as it takes from C to A
 Thus the whole time taken by the particle to return to
 $C = 2(t_1 + t_2)$

$$= 2 \left[\sqrt{\frac{a}{g}} \cdot \frac{2\pi}{3} + \sqrt{\frac{3a}{g}} \right] = \sqrt{\frac{a}{g}} \left[\frac{4\pi}{3} + 2\sqrt{3} \right].$$

6(c) Find the absolute value of the directional derivative of $\phi(x, y, z) = xyz^2$ at the point $(1, 1, 1)$ in the direction of the tangent to the curve $x = e^t$, $y = 2\sin t + 1$, $z = t - \cos t$ at $t=0$.

Soln:

Given $\phi(x, y, z) = xyz^2$

$$\begin{aligned}\text{Then grad } \phi &= \frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \mathbf{j} + \frac{\partial \phi}{\partial z} \mathbf{k} \\ &= yz^2 \mathbf{i} + xz^2 \mathbf{j} + 2xyz \mathbf{k}.\end{aligned}$$

$$\nabla \phi \Big|_{\text{at } (1, 1, 1)} = 2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}.$$

Also for the curve $x = e^t$, $y = 2\sin t + 1$
 $z = t - \cos t$

we have $\frac{dx}{dt} = e^t$, $\frac{dy}{dt} = 2\cos t$, $\frac{dz}{dt} = 1 + \sin t$

$$\left. \frac{dx}{dt} \right|_{t=0} = 1, \quad \left. \frac{dy}{dt} \right|_{t=0} = 2, \quad \left. \frac{dz}{dt} \right|_{t=0} = 1$$

now a vector along the tangent to the above curve at the point (x, y, z)

$$\begin{aligned}&= \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} + \frac{dz}{dt} \mathbf{k} \\ &= \mathbf{i} + 2\mathbf{j} + \mathbf{k}.\end{aligned}$$

If $\hat{a}$ be the unit vector in the direction of this tangent, then

$$\hat{a} = \frac{i+2j+k}{|i+2j+k|} = \frac{i+2j+k}{\sqrt{6}}$$

$\therefore$ The required directional derivative $= \hat{a} \cdot \text{grad } \phi$.

$$= \frac{i+2j+k}{\sqrt{6}} \cdot (2i+2j-2k)$$

$$= \frac{2+4-2}{\sqrt{6}}$$

$$= \frac{4}{\sqrt{6}}$$

The absolute value of the directional derivative is $= \left| \frac{4}{\sqrt{6}} \right|$
 $= \frac{4}{\sqrt{6}}$.

6(C)(ii) If $\nabla \cdot \vec{E} = 0$, $\nabla \cdot \vec{H} = 0$, $\nabla \times \vec{E} = -\frac{\partial \vec{H}}{\partial t}$ and $\nabla \times \vec{H} = \frac{\partial \vec{E}}{\partial t}$,
 then show that $\nabla^2 \vec{H} = \frac{\partial^2 \vec{H}}{\partial t^2}$ and $\nabla^2 \vec{E} = \frac{\partial^2 \vec{E}}{\partial t^2}$.

Solⁿ: $\nabla \times (\nabla \times \vec{E}) = \nabla \times \left(-\frac{\partial \vec{H}}{\partial t}\right) = -\frac{\partial}{\partial t} (\nabla \times \vec{H})$
 $= -\frac{\partial}{\partial t} \left(\frac{\partial \vec{E}}{\partial t}\right) = -\frac{\partial^2 \vec{E}}{\partial t^2}$

we know $\nabla \times (\nabla \times \vec{A}) = -\nabla^2 \vec{A} + \nabla (\nabla \cdot \vec{A})$

$\Rightarrow \nabla \times (\nabla \times \vec{E}) = -\nabla^2 \vec{E} + \nabla (\nabla \cdot \vec{E})$

then $\nabla^2 \vec{E} = \frac{\partial^2 \vec{E}}{\partial t^2}$

Similarly

$\nabla \times (\nabla \times \vec{H}) = \nabla \times \left(\frac{\partial \vec{E}}{\partial t}\right)$
 $= \frac{\partial}{\partial t} (\nabla \times \vec{E})$

$= -\frac{\partial}{\partial t} \left(-\frac{\partial \vec{H}}{\partial t}\right) = -\frac{\partial^2 \vec{H}}{\partial t^2}$

But $\nabla \times (\nabla \times \vec{H}) = -\nabla^2 \vec{H} + \nabla (\nabla \cdot \vec{H})$
 $= -\nabla^2 \vec{H}$

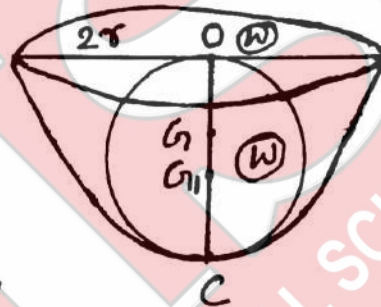
then $\nabla^2 \vec{H} = \frac{\partial^2 \vec{H}}{\partial t^2}$

7.(a)

A solid sphere rests inside a fixed rough hemispherical bowl of twice its radius. Show that, however large a weight is attached to the highest point of the sphere, the equilibrium is stable.

Solⁿ: Let r be the radius of the solid sphere which rests inside a fixed rough hemispherical bowl of radius $2r$.

Their point of contact is C and O is the highest point of the sphere so that $OC = 2r$.



Let W and w be weights of the sphere and the weight attached to the highest point of the sphere. The weight W of the sphere acts at the middle point G_1 of its diameter OC .

If h is the height of the centre of gravity of the combined body consisting of the sphere and the weight w attached to O , then

$$h = \frac{W \cdot r + w \cdot 2r}{W + w}$$

Here P_1 = the radius of curvature of the upper body at the point of contact
 C = the radius of the sphere = r ,

and P_2 = the radius of curvature of the lower body at the point of contact
 $C = -2r$, the negative sign is

taken because the surface of the lower fixed body i.e., the bowl at C is concave.

The equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{r_1} + \frac{1}{r_2} \quad \text{i.e., } \frac{1}{h} > \frac{1}{2r} \quad \text{i.e., } h < 2r$$

$$\text{i.e., } \frac{Wr + 2wr}{W + w} < 2r$$

$$\text{i.e., } Wr + 2wr < 2Wr + 2wr$$

$$\text{i.e., } Wr < 2Wr,$$

which is so whatever be the value of w .

Hence, however large a weight is attached to the highest point of the sphere, the equilibrium is stable.

7.(b) Verify Green's theorem in the plane for $\oint_C (xy + y^2) dx + x^2 dy$ where C is the closed curve of the region bounded by $y=x$ and $y=x^2$.

Solⁿ: By Green's theorem in plane,

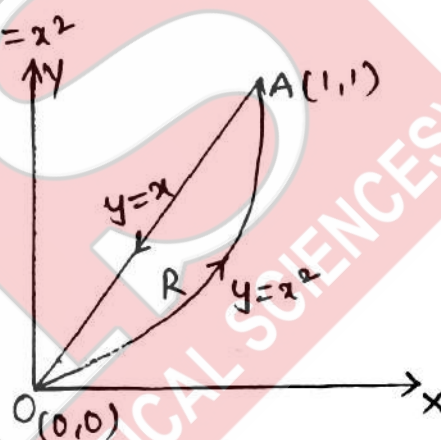
we have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \oint_C M dx + N dy$$

Here $M = xy + y^2$ and $N = x^2$

The curves $y=x$ and $y=x^2$ intersect at $(0,0)$ and $(1,1)$.

The +ve direction in traversing C is as shown in the figure.



$$\begin{aligned} \text{we have } \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy &= \iint_R \left[\frac{\partial}{\partial x} (x^2) - \frac{\partial}{\partial y} (xy + y^2) \right] dx dy \\ &= \iint_R (2x - x - 2y) dx dy \end{aligned}$$

$$= \iint_R (x - 2y) dx dy$$

$$= \int_{x=0}^1 \int_{y=x^2}^x (x - 2y) dy dx$$

$$= \int_{x=0}^1 (xy - y^2) \Big|_{y=x^2}^x dx$$

$$= \int_{x=0}^1 (x^2 - x^2 - x^3 + x^4) dx$$

$$= \int_{x=0}^1 (x^4 - x^3) dx = \left(\frac{x^5}{5} - \frac{x^4}{4} \right) \Big|_0^1 = \frac{1}{5} - \frac{1}{4} = -\frac{1}{20}$$

Now let us evaluate the line integral along C .

The line integral along C = line integral along $y=x^2$
 (from 0 to A)
 + line integral along $y=x$ (from A to 0).
 $= I_1 + I_2$

Along $y=x^2$; $dy=2x dx$

$$\begin{aligned}\therefore I_1 &= \int_0^1 [x(x^2) + x^4] dx + x^2(2x) dx \\ &= \int_0^1 (3x^3 + x^4) dx \\ &= \left[3 \frac{x^4}{4} + \frac{x^5}{5} \right]_0^1 \\ &= \frac{3}{4} + \frac{1}{5} = \frac{19}{20}\end{aligned}$$

Along $y=x$, $dy=dx$

$$\begin{aligned}\therefore I_2 &= \int_1^0 \{x(x) + x^2\} dx + x^2 dx \\ &= \int_1^0 3x^2 dx \\ &= \frac{3x^3}{3} = 0 - 1 = -1\end{aligned}$$

$$\therefore I_1 + I_2 = \frac{19}{20} - 1 = -\frac{1}{20}$$

Hence the theorem is verified.

7(c)(i) Find the general solution and singular solution of the differential equation

$$\left(1 + \frac{dy}{dx}\right)^3 = \frac{27}{8a} (x+y) \left(1 - \frac{dy}{dx}\right)^3.$$

Soln: $(1+p)^3 = \left(\frac{3}{2a}\right)^3 (x+y) (1-p)^3$

$$\frac{1+p}{1-p} = \frac{3}{2} \sqrt[3]{\frac{x+y}{a}} \quad \text{--- (1)}$$

$$p \left(1 + \frac{3}{2} \sqrt[3]{\frac{x+y}{a}}\right) = \frac{3}{2} \sqrt[3]{\frac{x+y}{a}} - 1$$

$$\frac{dy}{dx} = \frac{\frac{3}{2} \left(\frac{x+y}{a}\right)^{1/3} - 1}{\frac{3}{2} \left(\frac{x+y}{a}\right)^{1/3} + 1} \quad \text{--- (A)}$$

let $x+y=t \Rightarrow \frac{dt}{dx} = 1 + \frac{dy}{dx} = 1+p$

$$\frac{1+p}{1-p} = \frac{dt/dx}{2 - dt/dx} = \frac{3}{2} \left(\frac{x+y}{a}\right)^{1/3}$$

$$\frac{dt}{dx} = \frac{23}{2} \left(\frac{x+y}{a}\right)^{1/3} = 3 \left(\frac{x+y}{a}\right)^{1/3}$$

$$\frac{dx}{dt} = \frac{1}{3} \left(\frac{a}{x+y}\right)^{1/3} + \frac{1}{2} \Rightarrow \frac{dx}{dt} = \frac{1}{3} \left(\frac{a}{t}\right)^{1/3} + \frac{1}{2}$$

$$x = \frac{a^{1/3}}{3} \frac{t^{-1/3+1}}{2/3} + \frac{1}{2}t$$

$$\Rightarrow x = \frac{(x+y)^{2/3} \cdot a^{1/3}}{2} + \frac{(x+y)}{2} + C$$

$$x - \frac{x}{2} - \frac{y}{2} = \frac{(x+y)^{2/3} \cdot a^{1/3}}{2} + C$$

$$x-y = (x+y)^{2/3} a^{1/3} + c$$

(or) $(x-y+c)^3 = a(x+y)^2$ be general solution.

Singular solution:

$$F(x, y, c) = (x-y+c)^3 - a(x+y)^2 = 0$$

$$F = 0, \quad \frac{\partial F}{\partial c} = 0 \Rightarrow \frac{\partial F}{\partial c} = 3(x-y+c)^2 = 0$$

$$\Rightarrow c = y - x$$

$$\text{So, } 0 = (x-y+y-x) - a(x+y)^2$$

$$\Rightarrow x+y=0 \Rightarrow y=-x \text{ is envelope of general solution.}$$

$$\Rightarrow y = -x \text{ is singular solution.}$$

7(c)(ii) Find the complete solution of $x^3 \frac{d^3 y}{dx^3} + 3x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = x \log x$.

Sol'n: Given $(x^3 D^3 + 3x^2 D^2 + x D + 1)y = x \log x$, where $D \equiv \frac{d}{dx}$ — (1)

Let $x = e^z$ (or $z = \log x$) and $D_1 \equiv \frac{d}{dz}$, Then (1) becomes

$$[D_1(D_1 - 1)(D_1 - 2) + 3D_1(D_1 - 1) + D_1 + 1]y = ze^z$$

$$\Rightarrow (D_1^3 + 1)y = ze^z$$

Its auxiliary equation is $D_1^3 + 1 = 0$

$$\Rightarrow (D_1 + 1)(D_1^2 - D_1 + 1) = 0$$

$$\text{So that } D_1 = -1, (1 + i\sqrt{3})/2$$

$$\text{i.e. } D_1 = -1, \frac{1}{2} + i(\frac{\sqrt{3}}{2})$$

$$\therefore \text{C.F.} = C_1 e^{-z} + e^{z/2} [C_2 \cos\{(\sqrt{3}/2)z\} + C_3 \sin\{(\sqrt{3}/2)z\}]$$

$$= C_1 x^{-1} + x^{1/2} [C_2 \cos\{(\sqrt{3}/2) \log x\} + C_3 \sin\{(\sqrt{3}/2) \log x\}]$$

where C_1 and C_2 being arbitrary constants. as $x = e^z$

$$\text{P.I.} = \frac{1}{D_1^3 + 1} ze^z$$

$$= e^z \frac{1}{(D_1 + 1)^3 + 1} z$$

$$= \frac{1}{D_1^3 + 3D_1^2 + 3D_1 + 2} z$$

$$= e^z \frac{z}{2 \left[1 + \frac{D_1^3 + 3D_1^2 + 3D_1}{2} \right]}$$

$$= \frac{e^z}{2} \left[1 + \left(\frac{D_1^3 + 3D_1^2 + 3D_1}{2} \right) \right]^{-1/2}$$

$$= \frac{e^z}{2} \left[1 - \frac{(D_1^3 + 3D_1^2 + 3D_1)}{2} \right] z$$

$$= \frac{e^z}{2} \left[z - \frac{3}{2} \right]$$

$$= \frac{ze^z}{2} - \frac{3}{4}e^z$$

$$= \frac{e^z}{4} (2z - 3)$$

$$= \frac{\pi}{4} (2 \log x - 3)$$

Hence the required general solution is

$$y = C.P + P.I$$

$$y = C_1 x^{-1} + x^{1/2} \left[C_2 \cos\left(\frac{\sqrt{3}}{2} \log x\right) + C_3 \sin\left\{\left(\frac{\sqrt{3}}{2}\right) \log x\right\} \right. \\ \left. + \frac{\pi}{4} (2 \log x - 3) \right]$$

_____.

8.(a) Apply the method of variation of parameters to solve the equation $(x+2)y_2 - (2x+5)y_1 + 2y = (x+1)e^x$.

Sol'n: Rearranging the given equation

$$y_2 - \frac{(2x+5)}{x+2} y_1 + \frac{2}{x+2} y = \frac{(x+1)e^x}{x+2} \quad (1)$$

Comparing with $y_2 + P y_1 + Q = R$

$$P(x) = -\frac{(2x+5)}{x+2} \quad Q = \frac{2}{x+2} \quad R = \frac{(x+1)e^x}{(x+2)}$$

by observation $4 + 2P + Q = 0$.

e^{2x} is the part of complementary function.

By solving by the method of change of dependent variable when part of C.F. is known

$\therefore$ Let $u = e^{2x}$

$y = uv$ be the solution.

(1) transform to $\frac{dv}{dx} + \left(P + \frac{2}{u} \frac{du}{dx}\right) \frac{dv}{dx} = R/u$

$$\frac{dv}{dx} + \left(-\frac{2x+5}{x+2} + \frac{4e^{2x}}{e^{2x}}\right) \frac{dv}{dx} = 0$$

$$\frac{dv}{dx} + \left(\frac{2x+3}{x+2}\right) \frac{dv}{dx} = 0 \quad \frac{dv}{dx} = t$$

$$\frac{dt}{dx} + \left(2 - \frac{1}{x+2}\right)t = 0$$

integrating $\int \frac{dt}{t} + \int \frac{dx}{x+2} = 0$

$$\ln t + 2x - \ln(x+2) = C$$

$$\frac{dv}{dx} = C e^{-2x} (x+2)$$

$$v = \int C e^{-2x} (x+2) dx + C_2$$

$$v = c_1 \left(\frac{e^{-2x}(2x+5)}{-2} - \frac{e^{-2x}}{4} \right) + c_2$$

$$v = -\frac{c_1}{4} e^{-2x}(2x+5) + c_2$$

$$\therefore y_c = uv = -\frac{c_1}{4}(2x+5) + c_2 e^{2x}$$

$\therefore$ For solving variation of parameters

$$y_p = Au + Bv$$

$$\text{let } u = (2x+5) \quad v = e^{2x}$$

$$W(u, v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} 2x+5 & e^{2x} \\ 2 & 2e^{2x} \end{vmatrix} = e^{2x}(4x+8)$$

$$A(x) = \int \frac{Rv}{W} dx$$

$$= - \int \frac{(x+2)e^x}{(x+2)^2} \cdot \frac{e^{2x}}{e^{2x}(4x+8)} dx$$

$$= - \int \frac{(x+1)e^x}{4(x+2)^2} dx$$

$$= -\frac{1}{4} \int \left(\frac{e^x}{(x+2)} - \frac{e^x}{(x+2)^2} \right) dx \quad \left(\int e^x (f(x) + g(x)) dx = e^x f(x) \right)$$

$$A = -\frac{1}{4} \frac{e^x}{(x+2)}$$

$$B = \int \frac{Rv}{W} dx$$

$$= \int \frac{(2x+5) \cdot \frac{e^x(x+1)}{x+2}}{e^{2x}(4x+8)} dx$$

$$= \int \frac{(2x^2+7x+5)e^{-x}}{4(x+2)^2} dx$$

$$= \int \frac{2(x+2)^2 - (x+2) + 1}{4(x+2)^2} e^{-x} dx$$

$$= \int \frac{e^{-x}}{2} - \frac{1}{4} \int \left(\frac{1}{x+2} + \frac{1}{(x+2)^2} \right) e^{-x} dx$$

$$B = -\frac{e^{-x}}{2} + \frac{1}{4} \frac{e^{-x}}{x+2}$$

$$y_p = Au + Bv$$

$$y_p = -e^x$$

$$\boxed{y = y_c + y_p}$$

$$\boxed{y = c_1(2x+5) + c_2 e^{2x} - e^x}$$

Required solution.

8.(b) → verify divergence theorem for
 $F = (x^2yz)\hat{i} + (y^2xz)\hat{j} + (z^2xy)\hat{k}$. Taken over
 the rectangular parallelepiped $0 \leq x \leq a$,
 $0 \leq y \leq b$, $0 \leq z \leq c$.

Sol:- we have $\text{div } F = \nabla \cdot F$

$$= \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(y^2xz) + \frac{\partial}{\partial z}(z^2xy)$$

$$= 2x + 2y + 2z$$

$\therefore$ volume integral $= \iiint_V \nabla \cdot F \, dV$

$$= \iiint_V 2(x+y+z) \, dV$$

$$= 2 \int_{z=0}^c \int_{y=0}^b \int_{x=0}^a (x+y+z) \, dx \, dy \, dz$$

$$= 2 \iiint (x+y+z) \, dx \, dy \, dz$$

$$= 2 \int_{z=0}^c \int_{y=0}^b \left(\frac{x^2}{2} + yx + zx \right) \Big|_0^a \, dy \, dz$$

$$= 2 \int_{z=0}^c \int_{y=0}^b \left(\frac{a^2}{2} + ay + az \right) \, dy \, dz$$

$$= 2 \int_{z=0}^c \left(\frac{a^2 y}{2} + \frac{a y^2}{2} + azy \right) \Big|_0^b \, dz$$

$$= 2 \int_0^c \left(\frac{a^2 b}{2} + \frac{ab^2}{2} + abz \right) dz = 2 \left[\frac{a^2 b z}{2} + \frac{ab^2 z}{2} + a b \frac{z^2}{2} \right]_0^c$$

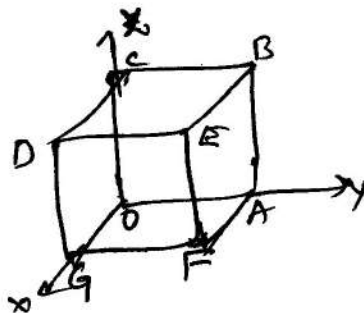
$$= abc(a+b+c)$$

Surface integral:

Now calculate $\iint_S F \cdot n \, ds$

over the face DEFG

$$n = \hat{i}, x = a$$



$$\begin{aligned} \therefore \iint_{DEFG} F \cdot n \, ds &= \int_{z=0}^c \int_{y=0}^b (a^2 - yz) \hat{i} + (yz - za) \hat{j} + (z^2 - ay) \hat{k} \cdot \hat{i} \, dy \, dz \\ &= \int_{z=0}^c \int_{y=0}^b (a^2 - yz) \, dy \, dz \\ &= \int_{z=0}^c \left(ay - \frac{y^2 z}{2} \right) \Big|_0^b \, dz \\ &= \int_{z=0}^c \left(ab - \frac{z b^2}{2} \right) \, dz = \left(abz - \frac{z^2 b^2}{4} \right) \Big|_0^c \\ &= abc - \frac{b^2 c^2}{4}. \end{aligned}$$

over the face ABCO, $n = -\hat{i}$, $x=0$.

$$\begin{aligned} \therefore \iint_{ABCO} F \cdot n \, ds &= \iint_{ABCO} [(0 - yz) \hat{i} + (yz) \hat{j} + z^2 \hat{k}] \cdot (-\hat{i}) \, dy \, dz \\ &= \int_{z=0}^c \int_{y=0}^b yz \, dy \, dz = \int_0^c \left(\frac{y^2 z}{2} \right) \Big|_0^b \, dz \\ &= \int_{z=0}^c \frac{b^2 z}{2} \, dz = \frac{b^2 c^2}{4}. \end{aligned}$$

over the face ADEF, $n = \hat{j}$, $y=b$.

$$\begin{aligned} \therefore \iint_{ADEF} F \cdot \hat{n} \, ds &= \int_{z=0}^c \int_{x=0}^a [(x^2 - bz) \hat{i} + (b^2 - zx) \hat{j} - (z^2 - bx) \hat{k}] \cdot \hat{j} \, dx \, dz \\ &= \int_{z=0}^c \int_{x=0}^a (b^2 - zx) \, dx \, dz = b^2 ca - \frac{a^2 c^2}{4} \end{aligned}$$

over the face OGDC, $n = \hat{j}$, $y=0$.

$$\therefore \iint_{OGDC} F \cdot \hat{n} \, ds = \int_{z=0}^c \int_{x=0}^a zx \, dx \, dz = \frac{ca^2}{4}.$$

over the face BCDE, $n = \hat{k}$, $z=c$.

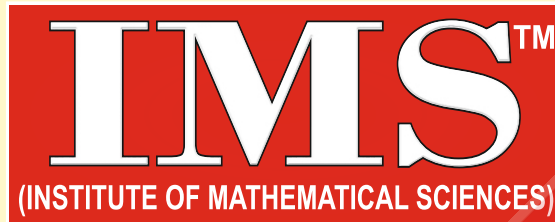
$$\therefore \iint_{BCDE} F \cdot n \, ds = \int_{y=0}^b \int_{x=0}^a (c^2 - xy) \, dx \, dy = c^2 ab - \frac{a^2 b^2}{4}$$

over the face AFGO, $n = -\hat{k}$, $z=0$.

$$\therefore \iint_{AFGO} F \cdot n \, ds = \int_{y=0}^b \int_{x=0}^a xy \, dx \, dy = -\frac{a^2 b^2}{4}.$$

Adding the six surface integrals, we get

$$\iint F \cdot n \, ds = \left(abc - \frac{b^2 c^2}{4} + \frac{c^2 b^2}{4} \right) + \left(b^2 ca - \frac{a^2 c^2}{4} + \frac{a^2 c^2}{4} \right) + \left(c^2 ab - \frac{a^2 b^2}{4} + \frac{a^2 b^2}{4} \right) \\ = (abc) \text{ (canceled)} \therefore \text{Hence the theorem is verified}$$



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