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(INSTITUTE OF MATHEMATICAL SCIENCES)

UPSC-CSE 2024

Mains

MATHEMATICS

Optional Paper-II

Solutions

1(a) Let G be a finite group of order mn, where m and n are prime numbers with m>n. Show that G has at most one subgroup of order m.

Sol^n: Suppose G has two subgroups H and K each of order m.

i.e. $O(H) = O(K) = m$

Now $m > n \Rightarrow m^2 > mn = O(G)$ (by hyp)

$\Rightarrow m > \sqrt{O(G)}$

i.e, $O(H) > \sqrt{O(G)}$ and $O(K) > \sqrt{O(G)}$

Since $HK \subseteq G$

$\therefore O(HK) \leq O(G)$

i.e, $O(G) \geq O(HK)$

$= \dfrac{O(H)\,O(K)}{O(H\cap K)}$

$> \dfrac{\sqrt{O(G)}\,\sqrt{O(G)}}{O(H\cap K)}$

$= \dfrac{O(G)}{O(H\cap K)}$

$\therefore O(G) > \dfrac{O(G)}{O(H\cap K)}$

$\Rightarrow O(H\cap K) > 1$

Since $H\cap K$ is a subgroup of H.

$\therefore$ By Lagrange's theorem,

$O(H\cap K)$ divides $O(H) = m$

i.e, $\dfrac{O(H)}{O(H\cap K)}$

$\Rightarrow O(H\cap K) = m$ $(\because O(H\cap K) > 1)$

$\Rightarrow O(H\cap K) = O(H)$

$\Rightarrow H\cap K = H \Rightarrow K = H$

$\therefore$ G has atmost one subgroup of order m.

1(b) If $w=f(z)$ is an analytic function of z, then show that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)|=0$.

Sol'n: we know that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)=4\frac{\partial^2}{\partial z\,\partial\bar{z}}$$

Hence $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\{\log|f'(z)|\}$

$$=4\frac{\partial^2}{\partial z\,\partial\bar{z}}\{\log|f'(z)|\}$$

$$=4\frac{\partial^2}{\partial z\,\partial\bar{z}}\left\{\frac{1}{2}\log|f'(z)|^2\right\}$$

$$=2\frac{\partial^2}{\partial z\,\partial\bar{z}}\left[\log\{f'(z)\,f'(\bar{z})\}\right]$$

$$=2\frac{\partial^2}{\partial z\,\partial\bar{z}}\left[\log f'(z)+\log f'(\bar{z})\right]$$

$$=2\frac{\partial}{\partial z}\left[\frac{f''(\bar{z})}{f'(\bar{z})}\right]=0$$

Since $f'(\bar{z})$ and $f''(\bar{z})$ are independent of z.

1.(c) Test the convergence of $\int_0^2 \frac{\log x}{\sqrt{2-x}}\,dx$.

Sol^n: Here $f(x) = \frac{\log x}{\sqrt{2-x}}$

clearly both 0 and 2 are points of infinite discontinuity of f on $[0,2]$. we may write

$$\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx \quad \text{---} \quad (1)$$

To test convergence of $\int_0^1 f(x)\,dx$ at $x=0$

Since $f(x)$ is negative on $(0,1]$, we consider $-f(x)$.

Take $g(x) = \frac{1}{x^n}$

$$\lim_{x\to 0+} \frac{-f(x)}{g(x)} = \lim_{x\to 0+} \frac{-x^n \log x}{\sqrt{2-x}} = 0 \text{ if } n>0$$

$\left[\because \lim_{x\to 0+} x^n \log x = 0 \text{ if } n>0\right]$

$\therefore$ Taking n between 0 and 1, the integral $\int_0^1 g(x)\,dx$ is convergent.

$\therefore$ By comparison test, $\int_0^1 -f(x)\,dx$ is also convergent.

To test the convergence of $\int_1^2 f(x)\,dx$ at $x=2$.

Take $g(x) = \frac{1}{\sqrt{2-x}}$

$$\lim_{x\to 2-} \frac{f(x)}{g(x)} = \lim_{x\to 2-} \log x = \log 2$$ which is non-zero and finite.

$\therefore$ By comparison test, $\int_1^2 f(x)\,dx$ and $\int_1^2 g(x)\,dx$ converge or diverge together.

But $\int_1^2 g(x)\,dx = \int_1^2 \frac{dx}{\sqrt{2-x}}$ | form $\int_a^b \frac{dx}{(b-x)^n}$

is convergent. $(\because n = \frac{1}{2} < 1)$

$\therefore \int_1^2 f(x)\,dx$ is also convergent.

Hence, from ①, $\int_0^2 f(x)\,dx$ is convergent.

1(d) If ϕ and φ are functions of x and y satisfying Laplace equation, then show that $f(z) = p+iq$, $i=\sqrt{-1}$ is an analytic function, where $p = \frac{\partial \phi}{\partial y} - \frac{\partial \phi}{\partial x}$ and $q = \frac{\partial \phi}{\partial x} + \frac{\partial \varphi}{\partial y}$.

Sol^n: Suppose $\phi(x,y)$ and $\varphi(x,y)$ satisfy Laplace's equation.

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \quad \text{--- (1)}$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{--- (2)}$$

Also suppose $p = \frac{\partial \phi}{\partial y} - \frac{\partial \phi}{\partial x}$, $q = \frac{\partial \phi}{\partial x} + \frac{\partial \varphi}{\partial y}$

To prove that $p+iq$

(i) $\frac{\partial p}{\partial x} = \frac{\partial q}{\partial y}$, $\frac{\partial p}{\partial y} = -\frac{\partial q}{\partial x}$

i.e. $\frac{\partial p}{\partial x} - \frac{\partial q}{\partial y} = 0$, $\frac{\partial p}{\partial y} + \frac{\partial q}{\partial x} = 0$.

and (ii) p_x, p_y, q_x, q_y all are continuous.

$$\frac{\partial p}{\partial x} - \frac{\partial q}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y} - \frac{\partial \varphi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x} + \frac{\partial \varphi}{\partial y}\right)$$

$$= \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y \partial x} - \frac{\partial^2 \varphi}{\partial y^2}$$

$$= -\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right) = 0, \text{ by (1)}$$

$$\therefore \frac{\partial p}{\partial x} - \frac{\partial t}{\partial y} = 0.$$

$$\frac{\partial p}{\partial y} + \frac{\partial q}{\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y} - \frac{\partial \varphi}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x} + \frac{\partial \varphi}{\partial y}\right)$$

$$= \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \varphi}{\partial y \partial x} + \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial x \partial y}$$

$$= \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial x^2} = 0, \text{ by (2)}$$

$$\therefore \frac{\partial p}{\partial y} + \frac{\partial q}{\partial x} = 0 \quad \text{--- (4)}$$

from (3) & (4), the result (i) follows:

Existence of (1) & (2) $\Rightarrow$ the result (ii).

1(e) Use two phase method to solve the following LPP:

$$\text{Max } Z = x_1 + 2x_2$$

s.c. $x_1 - x_2 \geq 3$

$2x_1 + x_2 \leq 10$

$x_1, x_2 \geq 0.$

Sol

$$\text{MAX } Z = x_1 + 2x_2 + 0S_1 + 0S_2 - MA$$

S.C.

$$x_1 - x_2 - S_1 + 0S_2 + A = 3$$

$$2x_1 + x_2 + 0S_1 + S_2 + 0A = 10$$

$x_1, x_2, S_1, S_2, A \geq 0.$

Phase-I

Let $\text{Max } Z^* = 0x_1 + 0x_2 + 0S_1 + 0S_2 - A.$

S.C.

$$x_1 - x_2 - S_1 + 0S_2 + A = 3$$

$$2x_1 + x_2 + 0S_1 + S_2 + 0A = 10.$$

$x_1, x_2, S_1, S_2, A \geq 0.$

IBFS: (x_1, x_2, S_1, S_2, A)

$= (0, 0, 0, 10, 3)$

$\boxed{\text{Max } Z^* = -3.}$

Let us construct simplex tables:

	c_j:	0	0	0	0	-1		
c_B	Basis	x_1	x_2	s_1	s_2	A	b	θ
-1	A	1	-1	-1	0	1	3	3 →
0	s_2	2	1	0	1	0	10	5
$z_j^* = \Sigma c_B a_{ij}$		-1	1	1	0	-1	-3	
c_j^*:	$c_j - z_j^*$	1	-1	-1	0	0		
0	x_1	1	-1	-1	0	1	3	
0	s_2	0	3	2	1	-2	4.	
$z_j^* = \Sigma c_B a_{ij}$		0	0	0	0	0	0	
c_j^*:	$c_j - z_j^*$	0	0	0	0	-1.	0	

$R_2 \to R_2 - 2R_1$

clearly all $c_j^* \leq 0$.

and $\max z^* = 0$.

Let us go to phase-II with original objective function subject to the same conditions:

	c_j:	1	2	0	0		
c_B	Basis	x_1	x_2	s_1	s_2	b	θ
1	x_1	1	-1	-1	0	3	—
0	s_2	0	3	2	1	4	4/3 →
z_j:	$\Sigma c_B a_{ij}$	1	-1	-1	0	3	
c_j:	$c_j - z_j$	0	2	1	0		
1	x_1	1	0	-1/3	1/3	13/3	
2	x_2	0	1	2/3	1/3	4/3	
z_j:	$\Sigma c_B a_{ij}$	1	2	1	1	17/3.	
c_j:	$c_j - z_j^*$	0	0	-1	-1		

$R_1 \to R_1 + R_2$

$-\frac{1}{3} + \frac{2}{3}$

$\therefore$ All c_j's ≤ 0

OBFS: $(x_1, x_2, s_1, s_2, A) = (13/3, 4/3, 0, 0, 0)$.

$\max z = 17/3.$

2(a) Using Cauchy's general principle of convergence, examine the convergence of the sequence $\langle f_n \rangle$, where $f_n = 1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}$

Sol'n: Here, for $n \geq m$, we have

$$|f_n - f_m| = \left|\left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{m!}+\frac{1}{(m+1)!}+\frac{1}{(m+2)!}+\cdots+\frac{1}{n!}\right) - \left(1+\frac{1}{1!}+\cdots+\frac{1}{m!}\right)\right|$$

$$= \frac{1}{(m+1)!}+\frac{1}{(m+2)!}+\cdots\frac{1}{n!}$$

$$= \frac{1}{2^m}+\frac{1}{2^{m+1}}+\cdots+\frac{1}{2^n}, \text{ as } n! \geq 2^{n-1} \ \forall n \in N$$

$$= \frac{1}{2^m}\left[1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{(n-m)+1}}\right] = \frac{1}{2^m} \times \frac{1-\left(\frac{1}{2}\right)^{n-m}}{1-\frac{1}{2}}$$

$$= \frac{1}{2^{m-1}} - \frac{1}{2^{n-1}} \leq \frac{1}{2^{m-1}}$$

$$\left[\because n \geq m \Rightarrow n-1 \geq m-1 \Rightarrow 2^{n-1} \geq 2^{m-1} \Rightarrow \frac{1}{2^{n-1}} \leq \frac{1}{2^{m-1}}\right]$$

Thus for $\varepsilon > 0$, $|f_n - f_m| < \varepsilon$ if $\frac{1}{2^{m-1}} < \varepsilon$ (or) $2^{m-1} > \frac{1}{\varepsilon}$

$$2^{m-1} > \frac{1}{\varepsilon} \Rightarrow (m-1)\log 2 > \log\left(\frac{1}{\varepsilon}\right)$$

$$\Rightarrow m > \left\{1+\log\left(\frac{1}{\varepsilon}\right)\times(\log 2)^{-1}\right\} \quad \text{—(1)}$$

for each $\varepsilon > 0$, there exists $m \in N$ such that

$|f_n - f_m| < \varepsilon \ \forall n \geq m$, where m is given by (1)

Hence $\langle f_n \rangle$ is a Cauchy sequence and so by Cauchy's criteria of convergence, $\langle f_n \rangle$ must be convergent sequence.

2(b) Show that every homomorphic image of an abelian group is abelian, but the converse is not necessarily true.

Sol^n: Let $(G,\cdot)$ be an abelian group and $(G',\cdot)$ be a group.

Let $f: G \to G'$ be a homomorphism and onto.

$\therefore G'$ is the homomorphic image of G i.e. $G' = f(G)$.

Let $a', b' \in G'$

$\therefore \exists$ elements $a, b \in G$ such that $f(a) = a'$ & $f(b) = b'$

Since G is abelian

$\therefore ab = ba$

$$\begin{aligned} \text{Now } a'b' &= f(a) f(b) \\ &= f(ab) \\ &= f(ba) \\ &= f(b) \cdot f(a) \\ &= b' \cdot a' \end{aligned}$$

$\therefore a'b' = b'a' \quad \forall\, a', b' \in G'$

$\therefore G'$ is an abelian.

The converse of the above theorem need not be true. i.e, If the homomorphic image of a group G is abelian. then the group need not be abelian.

Ex: P_3 is non-abelian group

A_3 is normal subgroup of P_3.

The quotient group $\frac{P_3}{A_3}$ is a homomorphic image of P_3.

Now P_3/A_3 is of order 2 and is abelian.

2.(c) Find the function which is analytic inside and on the circle $C: z = e^{i\theta}, 0 \le \theta \le 2\pi$ and has the value $\dfrac{(a^2-1)\cos\theta + i(a^2+1)\sin\theta}{a^4 - 2a^2\cos 2\theta + 1}$ on the circumference of C, where $a^2 > 1$.

Sol^n: Let $f(z)$ be the required function. Since $f(z)$ is analytic inside the circle $|z|=1$, it can be expanded in a Taylor's series at any point z inside this circle.

Thus $f(z) = \sum_0^\infty a_n z^n$ Taylor's Expansion

where $a_n = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)\,dz}{z^{n+1}}$ where C is the circle $|z|=1$.

Given that $f(z) = \dfrac{(a^2-1)\cos\theta + i(a^2+1)\sin\theta}{a^4 - 2a^2\cos 2\theta + 1}$ on $|z|=1$.

$$= \frac{a^2(\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta)}{a^4 - a^2(e^{2i\theta}+e^{-2i\theta})+1}$$

$$= \frac{a^2 e^{i\theta} - e^{-i\theta}}{a^4 - a^2(e^{2i\theta}+e^{-2i\theta}+1)}$$

$$= \frac{a^2 z - \frac{1}{z}}{a^4 - a^2\left(z^2 + \frac{1}{z^2}\right)+1}$$ since on $|z|=1, z=e^{i\theta}$

$$= \frac{z\left(a^2 - \frac{1}{z^2}\right)}{\left(a^2 - \frac{1}{z^2}\right)(a^2 - z^2)} = \frac{z}{a^2 - z^2}$$

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{z^{n+1}}\,dz$$

$$= \frac{1}{2\pi i}\int_C \frac{z}{a^2-z^2}\cdot\frac{1}{z^{n+1}}\,dz$$

$$= \frac{1}{2\pi i}\int_C \frac{1}{a^2-z^2}\cdot\frac{1}{z^n}\,dz$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{1}{a^2-e^{2i\theta}}\cdot\frac{1}{e^{in\theta}}\cdot e^{i\theta}\cdot i\,d\theta \quad \text{since } z=e^{i\theta}$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} e^{-i(n-1)\theta}\left[1-\frac{e^{2i\theta}}{a}\right]^{-1} d\theta$$

$$= \frac{1}{2\pi a^2}\int_0^{2\pi} e^{-i(n-1)\theta}\left[1+\frac{e^{2i\theta}}{a^2}+\frac{e^{4i\theta}}{a^4}+\cdots + \frac{e^{[2i(n-1)\theta]/2}}{a^{n-1}}+\cdots\right]d\theta$$

So $a_n = \frac{1}{2\pi a^2}\int_0^{2\pi}\frac{1}{a^{n-1}}\,d\theta$ if n is odd, all other integrals vanish being of the form $\int_0^{2\pi} e^{ik\theta}\,d\theta,\ k\neq 0.$

$$= \frac{1}{2\pi a^2}\cdot\frac{1}{a^{n-1}}\,2\pi = \frac{1}{a^{n+1}}$$

and $a_n=0$ if n is even, because then all the integrals vanish.

Hence $f(z) = \sum_0^\infty a_n z^n$ (n odd)

$$= \sum_0^\infty \frac{1}{a^{n+1}} z^n$$

$$= \frac{1}{a}\left[\frac{z}{a}+\frac{z^3}{a^3}+\frac{z^5}{a^5}+\cdots\right]$$

even values of n do not contribute.

$$= \frac{z}{a^2}\left[1+\frac{z^2}{a^2}+\frac{z^4}{a^4}\cdots\right]$$

$$= \frac{z}{a^2}\left[1-\frac{z^2}{a^2}\right]^{-1}$$

$$= \frac{z}{a^2}\cdot\frac{1}{1-\frac{z^2}{a^2}} = \frac{z}{a^2-z^2}$$

3(b) Consider the series $\sum_{n=1}^{\infty} U_n(x)$, $0 \le x \le 1$, the sum of whose first n terms is given by $S_n(x) = \frac{1}{2n^2}\log(1+n^4x^2)$, $x \in [0,1]$. Show that the given series can be differentiated term by term, though $\sum_{n=1}^{\infty} U_n'(x)$, does not converge uniformly on $[0,1]$

Sol^n: Here $U(x) = \lim_{n\to\infty} S_n(x)$

$$= \lim_{n\to\infty} \frac{\log(1+n^4x^2)}{2n^2} \left(\text{form } \frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} \frac{\frac{4n^3x^2}{1+n^4x^2}}{4n}$$

$$= \lim_{n\to\infty} \frac{n^2x^2}{1+n^4x^2} = 0 \text{ for } 0 \le x \le 1.$$

$$\therefore U'(x) = 0$$

Also $\lim_{n\to\infty} S_n'(x) = \lim_{n\to\infty}\left(\frac{1}{2n^2}\,\frac{2n^4x}{1+n^4x^2}\right)$

$$= \lim_{n\to\infty} \frac{n^2x}{1+n^4x^2} = 0 \text{ for } 0 \le x \le 1$$

$$\therefore U'(x) = \lim S_n'(x)$$

Thus term by term differentiable.

However, the series $\sum_{n=1}^{\infty} U_n'$ is not uniformly convergent for $0 \le x \le 1$. Since the sequence $\langle S_n' \rangle$ i.e, $\left\langle \frac{n^2x}{1+n^4x^2} \right\rangle$ has $x=0$ as a point of non-uniform convergence.

3.(c) Using duality principle, solve the following LPP:

$$\text{Min } Z = 4x_1 + 3x_2 + x_3$$

S.C.

$$x_1 + 2x_2 + 4x_3 \geq 12$$
$$3x_1 + 2x_2 + x_3 \geq 8$$
$$x_1, x_2, x_3 \geq 0$$

<u>sol</u>. Dual of the above is

$$\text{Max } W = 12w_1 + 8w_2$$

S.C.

$$w_1 + 3w_2 \leq 4$$
$$2w_1 + 2w_2 \leq 3$$
$$4w_1 + w_2 \leq 1$$
$$w_1, w_2 \geq 0$$

$$\Rightarrow \text{Max } W = 12w_1 + 8w_2 + 0S_1 + 0S_2 + 0S_3$$

S.C.

$$w_1 + 3w_2 + S_1 + 0S_2 + 0S_3 = 4$$
$$2w_1 + 2w_2 + 0S_1 + S_2 + 0S_3 = 3$$
$$4w_1 + w_2 + 0S_1 + 0S_2 + S_3 = 1$$
$$w_1, w_2, S_1, S_2, S_3 \geq 0$$

IBFS: $(w_1, w_2, S_1, S_2, S_3)$
$= (0, 0, 4, 3, 1)$

$\text{Max } W = 0$.

Let us write simplex tables!

	c_j:	12	8	0	0	0		
C_B	Basis	w_1	w_2	s_1	s_2	s_3	b	θ
0	s_1	1	3	1	0	0	4	4
0	s_2	2	2	0	1	0	3	$3/2$ →
0	s_3	4 ↑	1	0	0	1	1	$4/1$
w_j:	$\Sigma C_B a_{ij}$	0	0	0	0	0	0	
C_j:	$C_j - w_j$	12	8	0	0	0		
0	s_1	0	2	1	$-1/2$	0	$5/2$	$R_1 \to R_1 - R_2$
12	w_1	1	1	0	$1/2$	0	$3/2$	$R_3 \to R_3 - 4R_2$
0	s_3	0	-3	0	-2	1	-5	
w_j:	$\Sigma C_B a_{ij}$	12	12	0	6	0	18	$4 - 3/2 = 5/2$
C_j:	$C_j - w_j$	0	-6	0	-6	0		$1 - 4^2(3/2)$

clearly all C_j's ≤ 0
optimality has been obtained.

OBS: $(w_1, w_2, s_1, s_2, s_3)$
$= (3/2, 0, 5/2, 0, -5)$.

$\max W = 18$.

OBFS of primal problem is
$(x_1, x_2, x_3) = (0, 6, 0)$

$\text{Min } Z = 18$.

4(a) Consider the polynomial ring $Z[x]$ over the ring Z of integers. Let S be an ideal of $Z[x]$ generated by x. Show that S is prime but not a maximal ideal of $Z[x]$.

Sol^n: we have $S = \{x p(x) : p(x) \in Z[x]\}$.

Let $f(x) = a_0 + a_1 x + \cdots + a_m x^m$ and
$g(x) = b_0 + b_1 x + \cdots + b_n x^n$ be two polynomials in $Z[x]$ such that $f(x) g(x) \in S$. Then

$f(x) g(x) = x p(x)$, for some
$p(x) = c_0 + c_1 x + \cdots + c_r x^r \in Z[x]$

we have

$$(a_0 + a_1 x + \cdots)(b_0 + b_1 x + \cdots) = x(c_0 + c_1 x + \cdots)$$

Comparing the constant term on both the sides,
we get $a_0 b_0 = 0 \Rightarrow a_0 = 0$ (or) $b_0 = 0$ $(\because a_0, b_0 \in Z)$

If $a_0 = 0$, then $f(x) = a_1 x + a_2 x^2 + \cdots + a_m x^m$
$= x(a_1 + a_2 x + \cdots + a_m x^{m-1}) \in S$

Similarly, $b_0 = 0 \Rightarrow g(x) \in S$

Hence S is a prime ideal of $Z[x]$

However S is not a maximal ideal of $Z[x]$, since
$A = \{x f(x) + 2 g(x) : f(x), g(x) \in Z[x]\}$ is a proper ideal of $Z[x]$ such that

$$S \subset A \subset Z[x].$$

Notice that $2 \in A$ but $2 \notin S$ and $1 \in Z[x]$ but $1 \notin A$.

4(b) Find the upper and lower Riemann integrals for the function f defined on $[0,1]$ as follows:

$$f(x) = \begin{cases} (1-x^2)^{1/2}, & \text{if } x \text{ is rational} \\ (1-x), & \text{if } x \text{ is irrational} \end{cases}$$

Hence, show that f is not Riemann integrable on $[0,1]$.

Sol'n: Here $f(x) = \sqrt{(1-x^2)} = \sqrt{(1-x)(1+x)}$, if x is rational

and $f(x) = 1-x = \sqrt{(1-x)(1-x)}$, if x is irrational.

so $\sqrt{(1-x^2)} = \sqrt{(1-x)(1+x)} > \sqrt{(1-x)(1-x)}$ for $0<x<1$

Thus $(1-x^2)^{1/2} > 1-x$, for $0<x<1$.

Let M_r and m_r be the supremum and infimum of the given function f in I_r where I_r is the r^{th} usual sub-interval of any partition P of $[0,1]$.

Then, for all values of r, we have

$$M_r = (1-x^2)^{1/2} \text{ and } m_r = 1-x$$

Now, by definition, the upper Riemann integral

$$= \overline{\int_0^1} f(x)\,dx = \int_0^1 \sqrt{(1-x^2)}\,dx$$

$$= \left[\frac{1}{2} x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1 = \pi/4.$$

and the lower Riemann integral

$$= \underline{\int_0^1} f(x)\,dx = \int_0^1 (1-x)\,dx$$

$$= \left[x - \frac{1}{2}x^2\right]_0^1 = 1/2$$

Since upper and lower Riemann integrals are not equal, hence the given function f is not R-integrable on $[0,1]$.

4(c) The personnel manager of a company wants to assign officers A, B, and C to the regional offices at Delhi, Mumbai, Kolkota and Chennai. The cost of relocation (in thousand rupees) of the three officers at the four regional offices are given below:

Officer	Delhi	Mumbai	Kolkata	Chennai
A	16	22	24	20
B	10	32	26	16
C	10	20	46	30

Find the assignment which minimizes the total cost of relocation and also determine the minimum cost.

Sol Since the given assignment is not a square order matrix. Let us add dummy officer D row to get a 4×4 order matrix.

table(i)

	Delhi	Mumbai	Kolkata	Chennai
A	16	22	24	20
B	10	32	26	16
C	10	20	46	30
D	0	0	0	0

table(ii)

	Delhi	Mumbai	Kolkata	Chennai
A	0	6	8	4
B	0	22	16	6
C	0	10	36	20
D	0	0	0	0

since the minimum number of lines which cover all zero's $= 2 \neq$ order of assignment (4)

$\therefore$ optimal assignment shall not be made.

table(iii)

	Delhi	Mumbai	Kolkata	Chennai
A	0	2	4	0
B	0	18	12	2
C	0	6	32	16
D	4	0	0	0

since the minimum number of lines which cover all zero's = 3 ≠ order of assignment (4)
∴ optimal assignment shall not be made

table(iv)

	Delhi	Mumbai	Kolkata	Chennai
A	2	2	4	0
B	0	16	10	0
C	0	4	30	14
D	6	0	0	0

since the minimum number of lines which cover all zero's = 3 ≠ order of assignment (4) table (v)

	Delhi	Mumbai	Kolkata	Chennai
A	2	0	2	$\cancel{0}$
B	$\cancel{0}$	14	8	0
C	0	2	28	14
D	8	$\cancel{0}$	0	2

- Clearly the minimum number of lines which cover all zero's $= 4 =$ order of assignment
  ∴ optimal assignment shall be made.
- Every row and column contains at least one zero.
- Every row and column contains exactly one encircled zero.

∴ optimal assignment will be given:

A – Mumbai — 22
B – Chennai — 16
C – Delhi — 10
D – Kolkata — 0

Total minimum cost $= 48$ for the relocation

5.(b) Solve the following system of linear equations by Gauss-Jordan method :
$2x+3y-z=5$, $4x+4y-3z=3$ and $2x-3y+2z=2$.

Sol^n We can represent the given system of linear equations in matrix form as $AX=B$ where

$$A \equiv \begin{bmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ 2 & -3 & 2 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$$

Now to solve by Gauss-Jordan elimination method, we take augmented matrix as $[A|B]$.

$$\text{so } [A|B] \equiv \left[\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 4 & 4 & -3 & 3 \\ 2 & -3 & 2 & 2 \end{array}\right]$$

Now we need to convert it in row-reduced echelon form.

using partial pivoting, $R_2 \leftrightarrow R_1$ and $R_1 \to \frac{R_1}{4}$

$$\left[\begin{array}{ccc|c} 1 & 1 & -3/4 & 3/4 \\ 2 & 3 & -1 & 5 \\ 2 & -3 & 2 & 2 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & -3/4 & 3/4 \\ 0 & 1 & 1/2 & 7/2 \\ 0 & -5 & 7/2 & 1/2 \end{array}\right]$$

apply $R_3 \to R_3 + 5R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & -3/4 & 3/4 \\ 0 & 1 & 1/2 & 7/2 \\ 0 & 0 & 6 & 18 \end{array}\right]$$

apply $R_3 \to R_3/6$, $R_2 \to R_2 - R_3/2$, $R_1 \to R_1 + \frac{3}{4}R_3$, $R_1 \to R_1 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

as it is reduced to $[I|D]$ form we can state that the solution is $x=1$, $y=2$, $z=3$.

5. (c) (i) Determine the decimal equivalent in sign magnitude form of $(8D)_{16}$ and $(FF)_{16}$.

(ii) Determine the decimal equivalent of $(9B2.1A)_{16}$.

Sol^n

(i) The given numbers are in hexadecimal format as the base is 16. So to convert to decimal number form use 16 as

$$(8D)_{16} = 8\times 16^1 + (D)\times 16^0 \quad \text{where } D \text{ is } (13)_{10}$$
$$= 128 + 13$$
$$= (141)_{10}$$

Similarly

$$(FF)_{16} = F\times 16^1 + F\times 16^0 \quad \text{where } F \text{ is } (15)_{10}$$
$$= 240 + 15$$
$$= (255)_{10}$$

(ii) Decimal equivalent of hexadecimal number is found in 2 parts:

before decimal part:

$$(9B2)_{16} = 9\times 16^2 + B\times 16 + 2\times 16^0 \quad B \text{ is } (11)_{10}.$$
$$= 2304 + 176 + 2$$
$$= (2482)_{10}$$

after decimal part:

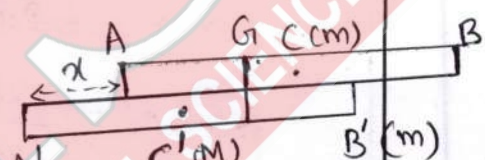
$$(1A)_{16} = 1\times\frac{1}{16} + A\times\frac{1}{16^2} \quad A \text{ is } (10)_{10}.$$
$$= 0.0625 + 0.0390625$$
$$= 0.1015625$$

So $(9B2.1A)_{16} = (2482.1015625)_{10}$ Ans.

5(d) A rough uniform board of mass m and length $2a$ rests on a smooth horizontal plane and a man of mass M walks on it from one end to the other. Find the distance covered by the board during this time.

Sol^n: Here the external forces are (i) the weights of the board and the man acting vertically downwards and (ii) the reaction of the horizontal plane acting vertically upwards. Thus there are no external forces in the horizontal direction,

$\therefore$ By D'Alembert's principle, the C.G. of the system will remain at rest. As a matter of fact as the man moves forward,



the board slips backwards, keeping the position of C.G. of the system unchanged.

Let AB be the position of the board when the man of mass M is at A.

$\therefore$ Distance of C.G. of the system from A (towards B)

$$= \frac{M.0 + m.AC}{M+m} = \frac{M.0 + m.a}{M+m} = \frac{ma}{M+m} = x_1 \text{ (say)} \quad (\because AG = BG = a)$$

Let $A'B'$ be the position of the board when the man reaches the other end B of the board. If the board slips through a distance $AA' = x$ (backwards) during the time the man walks from A to B, then in this position the distance of C.G. of the system from A (towards B)

$$= \frac{M.AB' + m.AC'}{M+m}$$

$$= \frac{M(2a-x) + m(a-x)}{M+m} = x_2 \text{ (Say)}$$

Since the position of the C.G. 'G' of the system remains unchanged.

$$\therefore x_1 = x_2$$

$$\Rightarrow \frac{ma}{M+m} = \frac{M(2a-x) + m(a-x)}{M+m}$$

$$\Rightarrow ma = 2aM + ma - (M+m)x$$

$$\Rightarrow x = \frac{2aM}{(m+M)}$$

which is the required distance.

5(Q) The velocity potential ϕ of a flow is given by
$$\phi = \frac{1}{2}(x^2 + y^2 - 2z^2).$$
Determine the streamlines.

Sol^n: We know that the velocity q of the fluid is given by
$$q = -\nabla\phi = -\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left\{\frac{1}{2}(x^2+y^2-2z^2)\right\}$$
$$\Rightarrow q = -\frac{1}{2}(2xi + 2yj - 4zk) \quad \text{--- (1)}.$$
But $q = ui + vj + wk$ --- (2)

Comparing (1) & (2) $u = -x$, $v = -y$, $w = +2z$

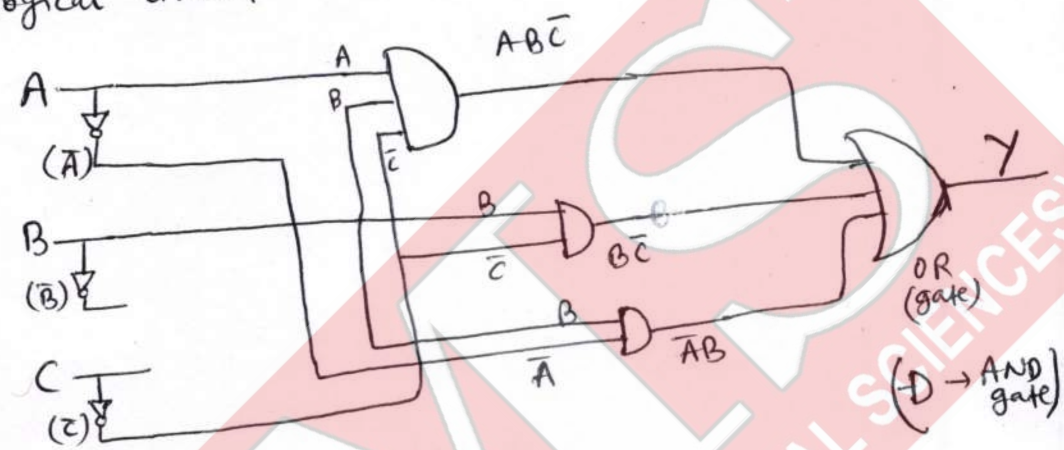
The equations of streamlines are given by
$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$
$$\frac{dx}{-x} = \frac{dy}{-y} = \frac{dz}{2z} \quad \text{--- (3)}$$
$$\Rightarrow \frac{2dx}{x} = \frac{2dy}{y} = \frac{dz}{-z}$$
Taking the first two fractions of (3), $\frac{1}{x}dx = \frac{1}{y}dy$

Integrating, $\log x = \log y + \log c_1 \Rightarrow x = c_1 y$ --- (4).

Taking the last two fractions of (3)
$$\frac{2}{y}dy + \frac{1}{z}dz = 0$$
Integrating; $2\log y + \log z = \log c_2$
$$\Rightarrow y^2 z = c_2 \quad \text{--- (5)}$$
(4) and (5) together give the equations of streamlines, c_1 and c_2 being arbitrary constants of integration.

6. (b) Draw the logical circuit for the Boolean expression $Y = AB\bar{C} + B\bar{C} + \bar{A}B$. Also, obtain the output Y (truth table) for the three input bit sequences: $A = 10001111$, $B = 00111100$, $C = 11000100$.

Sol^n We can ~~construct~~ use $Y = AB\bar{C} + B\bar{C} + \bar{A}B$ to draw logical circuit as



Now truth table for given input sequence

A	B	C	$AB\bar{C}$	$B\bar{C}$	$\bar{A}B$	Y
1	0	1	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	1	1	1
0	1	0	0	1	1	1
1	1	0	1	1	0	1
1	1	1	0	0	0	0
1	0	0	0	0	0	0
1	0	0	0	0	0	0

7. (b) Integrate $f(x) = 5x^3 - 3x^2 + 2x + 1$ from $x = -2$ to $x = 4$ using
(i) Simpson's 3/8 rule with width $h = 1$ and
(ii) Trapezoidal rule with width $h = 1$.

Sol^n Given $f(x) = 5x^3 - 3x^2 + 2x + 1$ also limit for integration $a = -2$, $b = 4$

So $I = \int_a^b f(x)\,dx$ we need to find this using Simpson's 3/8 rule.

given $h = 1$. so $nh = b - a$
$\Rightarrow n.1 = 4 - (-2) \Rightarrow n = 6$

Calculating values

x_i | $y_i = f(x_i)$
$x_0 = a = -2$ | $y_0 = f(x_0) = -55$
$x_1 = a+h = -1$ | $y_1 = f(x_1) = -9$
$x_2 = a+2h = 0$ | $y_2 = f(x_2) = 1$
$x_3 = 1$ | $y_3 = 5$
$x_4 = 2$ | $y_4 = 33$
$x_5 = 3$ | $y_5 = 115$
$x_6 = 4$ | $y_6 = 281$

now values calculated using $x_i = a + ih$, $y_i = f(x_i)$

Now formula of Simpson's 3/8th rule for integration is
$$I = \frac{3h}{8}\left[(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)\right]$$

Now putting the values
$$I = \frac{3.1}{8}\left[(-55 + 281) + 2(5) + 3(-9 + 1 + 33 + 115)\right]$$
$$= \frac{3}{8}(226 + 10 + 420) = 246$$

(ii) Now let us calculate with trapezoidal rule with $h=1$ so $nh = b-a \Rightarrow n \cdot 1 = 6$

$\Rightarrow n = 6$ (intervals)

$$I = \int_a^b f(x)dx = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+\cdots+y_{n-1})\right]$$

here

$x_0 = a = -2 \qquad y_0 = f(x_0) = -55$

$x_1 = a+h = -1 \qquad y_1 = f(x_1) = -9$

all other values same as calculated previously.

so now putting values directly into Trapezoidal formula for integration, we get

$$I = \frac{1}{2}\left[(-55+281) + 2(-9+1+5+33+115)\right]$$

$$= \frac{1}{2}\left[226 + 290\right] = \frac{516}{2}$$

$$= \underline{258}$$

7.(c) Show that the velocity field

$$u(x,y) = \frac{B(x^2-y^2)}{(x^2+y^2)^2}, \quad v(x,y) = \frac{2Bxy}{(x^2+y^2)^2}, \quad w=0$$

satisfies the equation of motion for an inviscid incompressible flow. Determine the pressure associated with this velocity field.

Sol'n: Euler's equation of motion in absence of external forces

is $\dfrac{dq}{dt} = -\dfrac{1}{\rho}\nabla p$

$$\Rightarrow \left(\frac{\partial}{\partial t} + q.\nabla\right) q = -\frac{1}{\rho}\nabla p$$

But motion is two dimensional as $w=0$ and $q = ui+vj$

$$\therefore \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right) q = -\frac{1}{\rho}\left(i\frac{\partial p}{\partial x} + j\frac{\partial p}{\partial y}\right)$$

putting the values.

$$\left[\frac{\partial}{\partial t} + \frac{B(x^2-y^2)}{(x^2+y^2)^2}\frac{\partial}{\partial x} + \frac{2Bxy}{(x^2+y^2)^2}\frac{\partial}{\partial y}\right](ui+vj) = -\frac{1}{\rho}\left(i\frac{\partial p}{\partial x} + j\frac{\partial p}{\partial y}\right)$$

As u, v are independent of t, by assumption.

$$\therefore \frac{\partial u}{\partial t} = 0 = \frac{\partial v}{\partial t}.$$

Hence the last gives

$$\frac{B}{(x^2+y^2)^2}\left[(x^2-y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}\right](ui+vj) = -\frac{1}{\rho}\left(i\frac{\partial p}{\partial x} + j\frac{\partial p}{\partial y}\right)$$

This $\Rightarrow$ $$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{B}{(x^2+y^2)^2}\left[(x^2-y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}\right]\frac{B(x^2-y^2)}{(x^2+y^2)^2} \quad \text{--- (1)}$$

and $$-\frac{1}{\rho}\frac{\partial p}{\partial y} = \frac{B}{(x^2+y^2)^2}\left[(x^2-y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}\right]\frac{2Bxy}{(x^2+y^2)^2} \quad \text{--- (2)}$$

But $$\frac{\partial}{\partial x}\left\{\frac{x^2-y^2}{(x^2+y^2)^2}\right\} = \frac{2x(3y^2-x^2)}{(x^2+y^2)^3} \quad \text{--- (3)}$$

$$\frac{\partial}{\partial y}\left\{\frac{x^2-y^2}{(x^2+y^2)^2}\right\} = \frac{-2y(3x^2-y^2)}{(x^2+y^2)^3} \quad \text{--- (4)}, \quad \frac{\partial}{\partial x}\left\{\frac{2y(y^2-x^2)}{(x^2+y^2)^3}\right\} \quad \text{--- (5)}$$

$$\frac{\partial}{\partial y}\left\{\frac{2xy}{(x^2+y^2)^2}\right\} = \frac{2x(x^2-y^2)}{(x^2+y^2)^3} \quad \text{--- (6)}$$

from equation (1), (3) & (4)

$$\frac{\partial p}{\partial x} = \frac{-2\rho B^2}{(x^2+y^2)^5}\left[(x^2-y^2)x(3y^2-x^2)-2xy^2(3x^2-y^2)\right]$$

$$\Rightarrow \frac{\partial p}{\partial x} = \frac{2\rho B^2 x}{(x^2+y^2)^3} \quad \text{--- (7)}$$

from (2), (5) & (6)

$$\frac{\partial p}{\partial x} = \frac{2\rho B^2}{(x^2+y^2)^5}\left[(x^2-y^2)y(y^2-x^2)+2x^2y(x^2-y^2)\right]$$

$$\Rightarrow \frac{\partial p}{\partial x} = \frac{2B^2 y}{(x^2+y^2)^4} \quad \text{--- (8)}$$

Differentiating (7) and (8) partially w.r.t y and x we find that

$$\frac{\partial^2 p}{\partial y\partial x} = \frac{\partial^2 p}{\partial x\partial y} \quad (\text{prove it})$$

this proves that velocity field satisfies the equation of motion

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy$$

Using (7) and (8)

$$dp = 2\rho B^2\left[\frac{x\,dx}{(x^2+y^2)^3} - \frac{y(x^2-y^2)}{(x^2+y^2)^4}dy\right]$$

$$= 2\rho B^2\left[M\,dx + N\,dy\right], \text{ say} \quad \text{--- (9)}$$

$$\frac{\partial M}{\partial y} = -\frac{6xy}{(x^2+y^2)^4} = \frac{\partial N}{\partial x}$$

$\therefore M\,dx + N\,dy$ is exact.

$$\int M\,dx + N\,dy = \int \frac{x\,dx}{(x^2+y^2)^3} + \int 0\,dy$$

$$= \frac{1}{2}\int 2x(x^2+y^2)^{-3}dx + C = -\frac{1}{4(x^2+y^2)^2} + C$$

In view of this (9) becomes.

$$p = -\frac{2\rho B^2}{4(x^2+y^2)^2} + C_1$$

$$\Rightarrow p = -\frac{\rho B^2}{2(x^2+y^2)^2} + C_1$$

This is the required expression for pressure.

8. (b) Using Newton's forward difference formula for interpolation, estimate the value of $f(2.5)$ from the following data :

x:	1	2	3	4	5	6
$f(x)$:	0	1	8	27	64	125

Soln: We are given with data for x & $y = f(x)$. We need to find $f(2.5)$ where 2.5 being on initial side of data, so using Newton's forward difference formula is better.

So first lets calculate cumulative frequency table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0 1	$y_0 = 0$					
		$\Delta y_0 = 1$	$\Delta^2 y_0 = 6$	6		
x_1 2	$y_1 = 1$				0	
		$\Delta y_1 = 7$	$\Delta^2 y_1 = 12$	6		0
x_2 3	$y_2 = 8$				0	
		$\Delta y_2 = 19$	$\Delta^2 y_2 = 18$	6		
x_3 4	$y_3 = 27$					
		$\Delta y_3 = 37$	$\Delta^2 y_3 = 24$			
x_4 5	$y_4 = 64$					
		$\Delta y_4 = 61$				
x_5 6	$y_5 = 125$					

Now we have, $x_0 = 1$, $x = 2.5$, $h = 1$

$$u = \frac{2.5 - 1}{1} = 1.5$$

Using Newton's forward Interpolation formula as

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$\because \Delta^4 y_0 = 0$ and so on. So terms till $\Delta^3 y_0$ taken

putting values

$$f(2.5) = 0 + 1.5(1) + \frac{1.5(0.5)}{2} \times 6^3 + \frac{1.5(0.5)(-0.5)}{6} \times 6$$

$$= 1.5 + 2.25 + (-0.375)$$

$$f(2.5) = 3.375 \quad \text{Ans.}$$

8(c) Suppose an infinite liquid contains two parallel, equal and opposite rectilinear vortices at a distance 2a. Show that the streamlines relative to the vortex are given by the equation

$$\log \frac{x^2+(y-a)^2}{x^2+(y+a)^2}+\frac{y}{a}=c$$

where c is a constant, the origin is the middle point of the join, and the line joining the vortices is the axis of y.

Sol^n: Let the vortices of strengths $+k, -k$ be at $A_1(-a,0)$, $A_2(a,0)$ s.t. A_1A_2 is along x-axis.

The complex potential due to this vortex pair at $P(x,y)$ is

$$W=\frac{ik}{2\pi}\log(z+a)-\frac{ik}{2\pi}\log(z-a)$$

(or) $\phi+i\psi=\frac{ik}{2\pi}\left[\log(y+a+ix)-\log(y-a+ix)\right]$

Equating imaginary parts from bothsides.,

$$\psi=\frac{k}{4\pi}\left[\log\{(y+a)^2+x^2\}-\log\{(y-a)^2+x^2\}\right] \quad \text{——(1)}$$

The vortex pair will move along a line parallel to y-axis with velocity

$$\frac{k}{2\pi A_1A_2}=\frac{k}{2\pi(2a)}=\frac{k}{4\pi a}$$

To reduce the system to rest, we have to superimpose a velocity $(-k/4\pi a)$ parallel to y-axis. If ψ' be the stream function due to this disturbance, Then

$$\frac{-k}{4\pi a} = v = -\frac{\partial \phi'}{\partial y} = \frac{\partial \psi'}{\partial x} \quad \therefore \psi' = -\frac{kx}{4\pi a}$$

The streamlines relative to the vortex system are given by ψ = Constant. i.e.

$$\frac{k}{4\pi}\left[\log\{(y+a)^2+x^2\} - \log\{(y-a)^2+x^2\}\right] - \frac{kx}{4\pi a} = \text{Const.}$$

$$\Rightarrow -\log\{(y+a)^2+x^2\} + \log\{(y-a)^2+x^2\} + \frac{y}{a} = \text{Const.} \quad (2)$$

$$\Rightarrow \log\left\{\frac{(y-a)^2+x^2}{(y+a)^2+x^2}\right\} + \frac{y}{a} = \text{constant.}$$

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