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(INSTITUTE OF MATHEMATICAL SCIENCES)

UPSC-CSE 2023

Mains

MATHEMATICS

Optional Paper-II

Solutions

1(5) Let G be a group of order 10 and G' be a group of order 6. Examine whether there exists a homomorphism of G onto G'.

Sol: Let G and G' be the groups such that $o(G) = 10$ and $o(G') = 6$.

If possible let $f: G \to G'$ be a homomorphism and onto.

i.e suppose there exists an epimorphism $f: G \to G'$.

$\therefore$ By fundamental theorem of homomorphism $\frac{G}{\ker f} \cong G'$

i.e $\phi: \frac{G}{K} \to G'$ is an isomorphism and onto.

$\therefore o\left(\frac{G}{K}\right) = o(G') = 6.$

$\Rightarrow \frac{o(G)}{o(K)} = 6$

$\Rightarrow \frac{10}{o(K)} = 6.$

This is absurd.

$\therefore G'$ is not a homomorphic image of G

i.e $f: G \to G'$ is not homomorphism and onto

i.e there does not exist any epimorphism $f: G \to G'$.

1(b) Express the ideal $4Z + 6Z$ as a principal ideal in the integral domain Z.

Sol: Let $Z = \{ ----3, -2, -1, 0, 1, 2, 3 --- \}$ be an integral domain.

Let 'S' be an ideal of Z

- If $S = \{0\}$ then it is a principal ideal generated by '0'
  there is nothing to prove.
- If $S \neq \{0\}$ then S contains non-zero elements.

Let $a (\neq 0)$ in S

$\Rightarrow -a \in S$ ($\because$ the ideal 'S' is additive subgroup of Z)

$\therefore$ S contains +ve and -ve integers.

Let s be the least +ve integer in 'S'

Let 'p' be any element in 'S'

$\therefore$ By division algorithm property, $\exists$ integers, q, r such that

$p = sq + r$; $(0 \leq r < s)$

now $q \in Z$, $s \in S \Rightarrow sq \in S$ & $qs \in S$ ($\because$ S is an ideal)

$\therefore p \in S$, $sq \in S$

$\Rightarrow p - sq \in S$

$\Rightarrow r \in S$ where $0 \leq r < s$

Which is a contradiction

$\therefore r = 0$

$p = sq$.

$\therefore p \in S \Rightarrow p = s\varepsilon$ for some $\varepsilon \in I$

$\therefore$ 'S' is a principal ideal of Z generated by s.

i.e $S = \langle s \rangle$.

$\therefore$ Every ideal of Z is a principal ideal

clearly $4Z = \{\ldots -4, 0, 4 \ldots\}$
$= \langle 4 \rangle = \{4x / x \in I\}$

and $6Z = \{\ldots -6, 0, 6 \ldots\}$
$= \langle 6 \rangle = \{6y / y \in I\}$

are principal ideals of Z.

$\therefore 4Z + 6Z = \{4x \pm 6y / x, y \in Z\}$
$= \{\ldots -4, -2, 0, 2, 4, 6, \ldots\}$
$= \langle 2 \rangle = \{2p / p \in Z\}$

is also principal ideal of Z.

1(C) → Test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)} \frac{x^{2n+1}}{2n+1}$$

Sol^n: Let $u_n = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)} \frac{x^{2n+1}}{2n+1}$

$$u_{n+1} = \frac{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)}{2\cdot 4\cdot 6\cdots(2n)(2n+2)} \frac{x^{2n+3}}{2n+3}$$

Now $\frac{u_n}{u_{n+1}} = \frac{2n+2}{2n+1} \cdot \frac{2n+3}{2n+1} \cdot \frac{1}{x^2}$

$$= \frac{\cancel{2n}\left[1+\frac{1}{n}\right]}{\cancel{2n}\left[1+\frac{1}{2n}\right]} \cdot \frac{\cancel{2n}\left[1+\frac{3}{2n}\right]}{\cancel{2n}\left[1+\frac{1}{2n}\right]} \cdot \frac{1}{x^2}$$

$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1\cdot 1}{1\cdot 1} \cdot \frac{1}{x^2} = \frac{1}{x^2}$

$\therefore$ By Ratio test $\sum u_n$ converges if $\frac{1}{x^2} > 1$ i.e. $x^2 < 1$ and diverges if $\frac{1}{x^2} < 1$ i.e. $x^2 > 1$

If $x^2 = 1$, then Ratio test fails.

when $x^2 = 1$,

we have $\frac{u_n}{u_{n+1}} = \frac{(2n+2)(2n+3)}{(2n+1)^2}$

$$= \frac{4n^2+10n+6}{4n^2+4n+1}$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left(\frac{4n^2+10n+6}{4n^2+4n+1}-1\right)$$

$$= \lim_{n\to\infty} \frac{6n^2+5}{4n^2+4n+1}$$

$$= \lim_{n\to\infty} \frac{6+\frac{5n}{n}}{4+\frac{4}{n}+\frac{1}{n^2}}$$

$$= \frac{6}{4} = \frac{3}{2} > 1$$

$\therefore$ By Raabe's Test, the series converges.

Hence $\sum u_n$ is convergent if $x^2 \le 1$ and divergent if $x^2 > 1$

1(d) State the sufficient condition for a function $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ to be analytic in its domain. Hence, show that $f(z) = \log z$ is analytic in its domain and find $\frac{df}{dz}$.

Solⁿ:- Let $f(z) = u(x,y) + iv(x,y)$ be defined in a domain D. The the sufficient condition for $f(z)$ to be analytic in its domain is that $u(x,y)$ and $v(x,y)$ have continuous partial derivatives which satisfy Cauchy - Riemann conditions

i.e. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

Given:-

$$f(z) = \log z$$
$$= \log (x+iy)$$

Putting $x = r\cos\theta$ and $y = r\sin\theta$

$$\Rightarrow r^2 = x^2+y^2 \Rightarrow r = \sqrt{x^2+y^2} \ \& \ \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\Rightarrow f(z) = \log(r\cos\theta + ir\sin\theta)$$
$$= \log r(\cos\theta + i\sin\theta)$$
$$= \log re^{i\theta}$$
$$= \log r + i\theta$$
$$= \log\sqrt{x^2+y^2} + i\tan^{-1}\left(\frac{y}{x}\right)$$

$\therefore u(x,y) = \log\sqrt{x^2+y^2}$ and $v(x,y) = -\tan^{-1}\left(\frac{y}{x}\right)$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{x^2+y^2}} \times \frac{1}{2\sqrt{x^2+y^2}}(2x) = \frac{x}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \frac{-y}{x^2} = \frac{-y}{x^2+y^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \frac{1}{x} = \frac{x}{x^2+y^2}$$

Clearly, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are continuous in $\mathbb{C}$.

Also, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\therefore$ The given function is analytic in its domain.

Now, $\frac{df}{dz} = \frac{d(\log z)}{dz} = \frac{1}{z}$.

1-(e) A person requires 24, 24 and 20 units of chemicals A, B and C respectively for his garden. Product P contains 2, 4, and 1 units of chemicals A, B and C respectively per jar and product Q contains 2, 1 and 5 units of chemicals A, B & C respectively per jar. If a jar of P costs ₹ 30 and a jar of Q costs ₹ 50, then how many jars of each should be purchased in order to minimize the cost and meet the requirements.

Sol^n

	P	Q	
A	2	2	24
B	4	1	24
C	1	5	20
Price →	30₹	50₹	

Given min req of A, B, C is 24, 24, 20 units.

In each Jar of P A, B, C is (2, 4, 1) units respectively. And for each Jar of Q A, B, C is (2, 1, 5) respectively.

Now let x jars of P & y Jars of Q is purchased such that Cost = $30x + 50y$ ₹. we have to minimise cost by fullfiling the chemical requirements. i.e

$\min Z = 30x + 50y$. with respect to conditions

$$2x + 2y \geq 24$$
$$4x + y \geq 24$$
$$x + 5y \geq 20$$

and $x, y \geq 0$

i.e The problem has been formulated as a minimization problem of LPP.

Let us solve this by graphical method.

So for that need to find intercepts & intersection points of each straight line represented by conditions.

let $2x + 2y = 24 \Rightarrow (x, 0) \equiv (12, 0)$, $(0, y) \equiv (0, 12)$

$4x + y = 24 \Rightarrow (x, 0) \equiv (6, 0)$, $(0, y) \equiv (0, 24)$

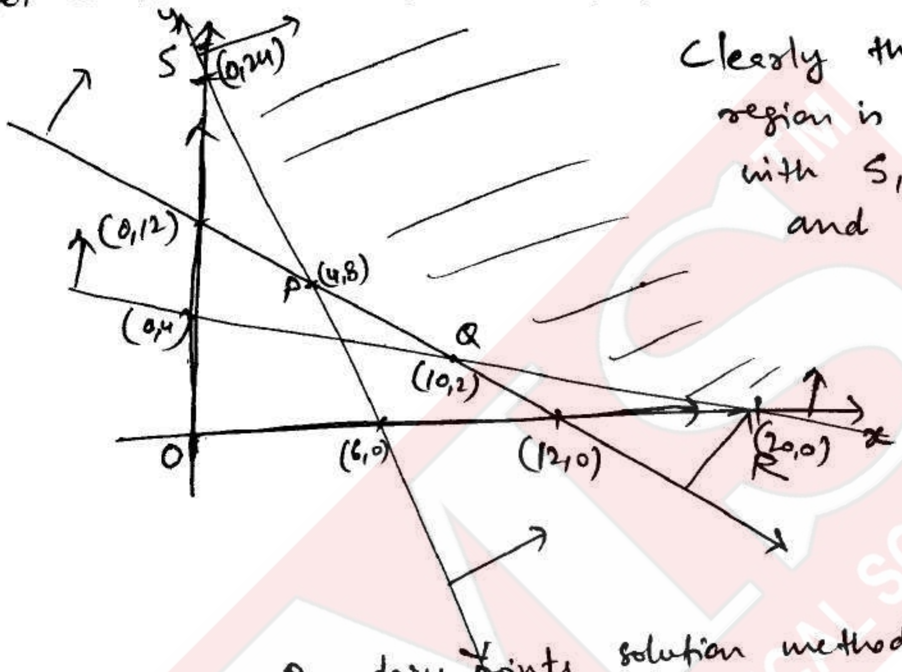
$x + 5y = 20 \Rightarrow (x, 0) \equiv (20, 0)$ $(0, y) \equiv (0, 4)$

Intersection of $2x+2y=24$ & $4x+y=24$ is $(x,y) \equiv (4,8)$

Intersection of $2x+2y=24$ & $x+5y=20$ is $(x,y) \equiv (10,2)$

Intersection of $4x+y=24$ & $x+5y=20$ is $(x,y) \equiv$

Let us draw the regions on graph from condition:



Clearly the shaded region is common with S, P, Q, R and above.

Now using Boundary points solution method $S(0,24)$ $P(4,8)$ $Q(10,2)$ $R(20,0)$ can be probable solution points.

$\text{Min } Z = 30x + 50y$

at $S \equiv 0 + 1200 = 1200$ ₹.

at $P \equiv 120 + 400 = 520$ ₹

at $Q \equiv 300 + 100 = 400$ ₹

at $R = 600 + 0 = 600$ ₹

Clearly we got min Z at Q which is 400 ₹ for purchase of P type 10 Jars & Q type 2 Jars.

Ans

2(a) → Prove that a non-commutative group of order $2p$, where 'p' is an odd prime must have a subgroup of order 'p'.

Sol) Let $(G, \times)$ be a non-commutative group such that $o(G) = 2p$. —— (1)
where 'p' is an odd prime.

$\therefore p > 2$.

To prove that G must have a subgroup of order 'p'.

Since G is a finite group; $a \in G$

$\therefore \dfrac{o(G)}{o(a)}$

$\Rightarrow \dfrac{2p}{o(a)}$

$\Rightarrow o(a) = 1, 2, p, 2p$.

Case (i) when $o(a) = 1$:

$\therefore a^1 = e$

$\therefore a = e$.

$\therefore H = \{e\}$ is an improper subgroup of G

s.t $o(H) = 1 \neq p$

Case (ii): when $o(a) = 2$

$\therefore a^2 = e \Rightarrow a^{-1} = a$.

Let $a, b \in G \Rightarrow ab \in G$.

$\therefore (ab)^2 = e$.

$(ab)(ab) = e$

$\Rightarrow a(ba)b = e$

$\Rightarrow ba = a^{-1}b^{-1} = ab$.

$\therefore$ $ba = ab$ $\forall a, b \in G$

which is contradiction to the fact that G is non-commutative.

$\therefore$ $o(a) \neq 2$.

case(iii) when $o(a) = 2p$:

$\therefore$ $o(a) = 2p = o(G)$

$\therefore$ $G = \langle a \rangle$

$\therefore$ G is cyclic group.

Since every cyclic group is an abelian

which is contradiction to the fact that G is not an abelian.

$\therefore$ $o(a) \neq 2p$

case(iv) when $o(a) = p$.

$\therefore$ $a^p = e$. — (A)

Let $H = \{e, a, a^2, \ldots a^{p-1}\} \subseteq G$.

clearly $H \leq G$ as

$a^i, a^j \in H$ $0 \leq i \neq j \leq p-1$.

$\Rightarrow a^i . a^j = a^{i+j}$

$= a^{pq+r}$ $\quad (\because i+j = pq+r$, $0 \leq r < p)$

$= a^{pq} . a^r$

$= (a^p)^q . a^r$

$= e^q . a^r$

$= a^r \in H.$ $\quad (\because e^q = e)$

$\therefore$ $H < G$ and

Clearly $o(H) = p$

because:

if possible let $a^i = a^j$; $0 \le j < i < p$

$\Rightarrow a^i . a^{-j} = a^j a^{-j}$ $(\because a^j \in G \Rightarrow a^{-j} \in G)$

$\Rightarrow a^{i-j} = a^0 = e.$

$\Rightarrow a^{i-j} = e$ where $0 < i-j < p$.

This contradicts (A).

$\therefore o(H) = p < 2p = o(G)$

$\therefore H$ is a subgroup of order p.

$\therefore G$ must have a subgroup H of order p.

2(b) Using the method of Lagrange's multipliers, find the minimum and maximum distances of the point $P(2,6,3)$ from the sphere $x^2+y^2+z^2=4$.

Solⁿ: Let $Q(x,y,z)$ be any point on the sphere $P(2,6,3)$ the given point.

$\therefore PQ^2 = (x-2)^2+(y-6)^2+(z-3)^2 = f(x,y,z)$ say — (1)

we have to find the maximum and minimum values of $f(x,y,z)$ subject to the condition

$$\phi(x,y,z) = x^2+y^2+z^2-4 = 0 \quad \text{— (2)}$$

Now let us consider the function F of independent variables x, y, z

$$\therefore F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$$

$$= (x-2)^2+(y-6)^2+(z-3)^2 + \lambda(x^2+y^2+z^2-4)$$

$$dF = [2(x-2)+2x\lambda]dx + [2(y-6)+2y\lambda]dy + [2(z-3)+2z\lambda]dz = 0$$

At stationary points $dF=0$

$\therefore F_x = 0 \Rightarrow 2(x-2)+2\lambda x = 0$

$\Rightarrow (x-2)+\lambda x = 0$ — (3)

$F_y = 0 \Rightarrow (y-6)+\lambda y = 0$ — (4)

$f_z = 0 \Rightarrow (z-3) + \lambda z = 0$ ⟵ (5)

from (3), (4) & (5)

$$-\frac{(x-2)}{x} = -\frac{(y-6)}{y} = -\frac{(z-3)}{z} = \lambda$$

$$= \pm\frac{\sqrt{(x-2)^2+(y-6)^2+(z-4)^2}}{\sqrt{x^2+y^2+z^2}}$$

$$= \pm\frac{\sqrt{f}}{2}$$

$$\therefore \lambda = \pm\frac{\sqrt{f}}{2}$$

Substituting λ in (3), (4) & (5), we get

(3) ≡ $x = \frac{2}{1+\lambda} = \frac{2}{1\pm\sqrt{f}/2}$

$y = \frac{6}{1+\lambda} = \frac{6}{1\pm\frac{\sqrt{f}}{2}}$

$z = \frac{3}{1+\lambda} = \frac{3}{1\pm\frac{\sqrt{f}}{2}}$

(2) ≡ $x^2+y^2+z^2=4$

$$\Rightarrow \frac{4+36+9}{\left(1\pm\frac{\sqrt{f}}{2}\right)^2} = 4$$

$$\Rightarrow 49 = 4\left(1\pm\frac{\sqrt{f}}{2}\right)^2$$

$$= (2\pm\sqrt{f})^2$$

$$\Rightarrow 2\pm\sqrt{f} = \pm 7$$

$$\Rightarrow \sqrt{f} = 5, +9$$

Hence maximum $PQ = 81$

minimum $PQ = 25$

2.(c) Evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\, d\theta$ using contour integration.

Sol^n: If $z = e^{i\theta}$ then $z^2 = e^{i2\theta} = \cos 2\theta + i\sin 2\theta$.

let us take the curve as circle C with radius 1 so $|z|=1$

$z = e^{i\theta}$ for $\theta \in [0, 2\pi]$.

Then $$I = \int_0^{2\pi} \frac{R.P.\, e^{i2\theta}}{5+4\frac{(e^{i\theta}+e^{-i\theta})}{2}}\, d\theta$$ $\because \cos 2\theta$ is real part of $e^{i2\theta}$.

$$I = R.P. \oint_C \frac{z^2}{5+2(z+z^{-1})} \frac{dz}{iz}$$ for $|z|=1$ $z\bar{z}=1$, $\cos\theta = \frac{z+\bar{z}}{2}$

$$I = R.P \oint_C \frac{z^2\, dz}{2i\cancel{z}\frac{(\frac{5}{2}z+z^2+1)}{\cancel{z}}} = R.P \oint_C \frac{z^2\, dz}{2i(z^2+\frac{5}{2}z+1)}$$

Now this integral can be evaluated using Cauchy's Residue theorem. $I = 2\pi i \sum R^+$ where $\sum R^+$ is sum of all residues of $f(z)$ with $|z|=1$ circle curve.

so need to find poles at which residues to be calculated. for Poles put $z^2 + \frac{5}{2}z + 1 = 0$

$$\text{or } z = \frac{-\frac{5}{2} \pm \sqrt{\frac{25}{4}-4}}{2} = \frac{-\frac{5}{2} \pm \frac{3}{2}}{2} = \frac{-5\pm 3}{4}$$

or $z = -\frac{8}{4}, -\frac{2}{4}$ or $z = -2, -\frac{1}{2}$.

clearly $z=-2$ lie outside of circle. so only $z = -\frac{1}{2}$ is simple pole of it.

a_{-1} is residue at simple pole $z=-\frac{1}{2}$ is

$$a_{-1} = \lim_{z\to -\frac{1}{2}} \frac{\cancel{(z+\frac{1}{2})}\, z^2}{2i\cancel{(z+\frac{1}{2})}(z+2)} = \frac{-i}{2}\,\frac{(\frac{1}{4})}{\frac{3}{2}}$$
$$= \boxed{\frac{-i}{12}}$$

So $I = \oint_C \frac{R.P.\ z^2\, dz}{2i\left(z^2+\frac{5}{2}z+1\right)} = 2\pi i \sum R^+$

$= 2\pi i \left[\frac{-i}{12}\right]$

$= \frac{\pi}{6}$

$\because$ I has real part only so $I = \frac{\pi}{6}$

or $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\, d\theta = \frac{\pi}{6}$ Ans.

3(a) Prove that x^2+1 is an irreducible polynomial in $Z_3[x]$. further show that the quotient ring $\frac{Z_3[x]}{\langle x^2+1\rangle}$ is a field of 9 elements.

Sol^n: we have $Z_3 = \{0, 1, 2\}$.

we observe that $f(x) = x^2+1$ is not satisfied by the elements of Z_3, (i.e., $f(\alpha) \neq 0$ for each $\alpha \in Z_3$)

$$\left[\begin{array}{l} \because f(0) = 1 \\ f(1) = 2 \\ f(2) = 5 = 2 \end{array}\right]$$

Consequently, x^2+1 is not expressible as a product of two linear factors in $Z_3[x]$.

Hence x^2+1 is irreducible over Z_3

By known theorem

$\frac{Z_3[x]}{x^2+1}$ is a field.

Any element of this field is of the form $f(x) + \langle x^2+1\rangle$ where $f(x) \in Z_3[x]$.

By division algorithm in $Z_3[x]$, there exists $t(x), r(x) \in Z_3[x]$ such that

$f(x) = t(x)(x^2+1) + r(x)$, where
$r(x) = 0$ or $\deg r(x) < \deg(x^2+1) = 2$,

we may take $r(x) = \alpha x + \beta \in Z_3[x]$.

Since $t(x)(x^2+1) \in \langle x^2+1 \rangle$

therefore $f(x) + \langle x^2+1 \rangle = r(x) + \langle x^2+1 \rangle$

$= \alpha x + \beta + \langle x^2+1 \rangle$ — (1)

In the above expression $\alpha, \beta \in Z_3$

and $O(Z_3) = 3$.

Consequently, each α and β can be selected in 3 ways. Hence, by (1) the no. of elements of the field

$\frac{Z_3[x]}{\langle x^2+1 \rangle}$ is $3^2 = 9$.

3.(b) Prove that $u(x,y) = e^x(x\cos y - y\sin y)$ is harmonic. Find its conjugate harmonic function $v(x,y)$ and express the corresponding analytic function $f(z)$ in terms of z.

Sol^n Any function $f(x,y)$ is said to be harmonic if $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$.

here given $u(x,y) = e^x(x\cos y - y\sin y)$.

$$\frac{\partial u}{\partial x} = e^x(x\cos y - y\sin y) + e^x(\cos y)$$

$$\frac{\partial^2 u}{\partial x^2} = e^x(x\cos y - y\sin y) + e^x(\cos y) + e^x(\cos y)$$

$$\frac{\partial u}{\partial y} = e^x(-x\sin y - \sin y - y\cos y)$$

$$\frac{\partial^2 u}{\partial y^2} = e^x(-x\cos y - \cos y - \cos y + y\sin y)$$

Now let us check $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

$$= \cancel{e^x x\cos y} - \cancel{e^x y\sin y} + \cancel{2e^x\cos y} - \cancel{e^x x\cos y}$$
$$- \cancel{2e^x\cos y} + \cancel{e^x y\sin y} = 0$$

i.e $u(x,y)$ is harmonic.

The conjugate harmonic function of $v(x,y)$ is $v(x,y)$ such that $f(z) = u(x,y) + i\,v(x,y)$ is analytic function. where $v(x,y)$ is harmonic too.

$\because$ $f(z)$ to analytic it must satisfy Cauchy-Riemann equations i.e $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

so $e^x(x\cos y - y\sin y) + e^x\cos y = \frac{\partial v}{\partial y}$

or $\int \partial v = \int e^x(x\cos y - y\sin y)\,dy + \int e^x\cos y\,dy$

So $v(x,y) = e^x x\sin y - \int e^x y\sin y\, dy + e^x \sin y + f(x)$

where $f(x)$ is function of x.

$\int y\sin y\, dy = -y\cos y + \int \cos y\, dy = -y\cos y + \sin y$

So $v(x,y) = e^x x\sin y + e^x y\cos y - e^x \cancel{\sin y} + e^x \cancel{\sin y} + f(x)$

now $\frac{\partial v}{\partial x} = e^x x\sin y + e^x\sin y + e^x y\cos y + f'(x)$.

Now $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow -\cancel{e^x x\sin y} - \cancel{e^x\sin y} - \cancel{e^x y\cos y} =$

$-\cancel{e^x x\sin y} - \cancel{e^x \sin y} - \cancel{e^x y\cos y} - f'(x)$.

or $f'(x) = 0 \Rightarrow f(x) = c_1$ (constant).

So $v(x,y) = e^x(x\sin y + y\cos y) + c_1$ which is harmonic conjugate of $u(x,y)$ and makes $f(z) = u(x,y) + iv(x,y)$ analytic.

Now need to find $f(z)$ in terms of 'z'.

So for that let us use Milne-Thomson's Method:

for which $x = \frac{z+\bar{z}}{2}$ & $y = \frac{z-\bar{z}}{2i}$ taken & replaced in u & v. Now taking $z = \bar{z}$ we get $x = z, y = 0$

and we know that $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ (By Cauchy Riemann Eq^n)

So $f'(z) = \left(e^x x\cos y - e^x y\sin y + e^x\cos y\right)\Big|_{x=z,y=0} + i\left(-xe^x\sin y - e^x\sin y - e^x y\cos y\right)$ at $x=z$, $y=0$.

$f'(z) = e^z z + e^z + i(-z\cdot 0 - e^z\cdot 0 - 0)$

So $f(z) = \int (e^z z + e^z)\, dz + C_2$ $\quad C_2$ is any complex const

$= e^z + e^z(z-1) + C$

$$\boxed{f(z) = ze^z + C}$$

is required $f(z)$ in terms of 'z'. Ans

3(c) Solve the following linear programming problem by ~~use~~ Big M method:

Minimize $Z = 2x_1 + 3x_2$

subject to $x_1 + x_2 \geq 9$

$x_1 + 2x_2 \geq 15$

$2x_1 - 3x_2 \leq 9$

$x_1, x_2 \geq 0$

Is the optimal solution unique? Justify your answer.

Sol^n:- The given problem is of minimization type. Let us convert this into maximization type.

Let Max $z^* = \text{Min}(-z) = -2x_1 - 3x_2$

subject to $x_1 + x_2 \geq 9$

$x_1 + 2x_2 \geq 15$

$2x_1 - 3x_2 \leq 9$

$x_1, x_2 \geq 0$

Introducing slack, surplus and artificial variables to convert the given problem into standard form.

Max $z^* = -2x_1 - 3x_2 + 0S_1 + 0S_2 + 0S_3 - MA_1 - MA_2$

subject to $x_1 + x_2 - S_1 + 0S_2 + 0S_3 + A_1 + 0A_2 = 9$

$x_1 + 2x_2 + 0S_1 - S_2 + 0S_3 + 0A_1 + A_2 = 15$

$2x_1 - 3x_2 + 0S_1 + 0S_2 + S_3 + 0A_1 + 0A_2 = 9$

$x_1, x_2, S_1, S_2, S_3, A_1, A_2 \geq 0$

No. of variables $= 7$

No. of equations $= 3$

$\therefore$ No. of free variables $= 7 - 3 = 4$

Initial basic feasible solution:-

$(x_1, x_2, S_1, S_2, S_3, A_1, A_2) = (0, 0, 0, 0, 9, 9, 15)$

and value of z^* at this solution is $-24M$

Let us draw simplex table and use Big-M method to solve the given problem.

	c_j	-2	-3	0	0	0	$-M$	$-M$			
C_B	Basic	x_1	x_2	S_1	S_2	S_3	A_1	A_2	b	θ	
$-M$	A_1	1	1	-1	0	0	1	0	9	9	
$-M$	A_2	1	2	0	-1	0	0	1	15	$15/2 \rightarrow$	
0	S_3	2	-3	0	0	1	0	0	9	-3	
$z_j=\sum c_B a_{ij}$		$-2M$	$-3M$	M	M	0	$-M$	$-M$	$-24M$		
$C_j=c_j-z_j$		$-2+2M$	$-3+3M \uparrow$	$-M$	$-M$	0	0	0			
$-M$	A_1	$1/2$	0	-1	$1/2$	0	1	$-1/2$	$3/2$	$3 \rightarrow$	
-3	x_2	$1/2$	1	0	$-1/2$	0	0	$1/2$	$15/2$	15	$R_2 \to \frac{1}{2}R_2$
0	S_3	$7/2$	0	0	$-3/2$	1	0	$3/2$	$63/2$	9	$R_1 \to R_1 - R_2$
$z_j=\sum c_B a_{ij}$		$-\frac{M}{2}-\frac{3}{2}$	-3	$-M$	$-\frac{M}{2}+\frac{3}{2}$	0	$-M$	$-\frac{M}{2}-\frac{3}{2}$	$\frac{-3M-45}{2}$		$R_3 \to R_3 + 3R_2$
$C_j=c_j-z_j$		$-\frac{1}{2}+\frac{M}{2} \uparrow$	0	M	$\frac{M}{2}-\frac{3}{2}$	0	0	$-\frac{M}{2}+\frac{3}{2}$			
-2	x_1	1	0	-2	1	0	2	-1	3		
-3	x_2	0	1	1	-1	0	-1	1	6		$R_1 \to 2R_1$
0	S_3	0	0	7	-5	1	-7	2	21		$R_2 \to R_2 - \frac{1}{2}R_1$
$z_j=\sum c_B a_{ij}$		-2	-3	1	1	0	-1	-1	-24		$R_3 \to R_3 - \frac{7}{2}R_1$
$C_j=c_j-z_j$		0	0	-1	-1	0	$-M+1$	$-M+1$			

$\because$ All C_j's ≤ 0

$\therefore$ Optimality has been attained.

~~Opti~~ Max $z^* = -24$

$\Rightarrow$ Min $z = -(-24) = 24$

$\therefore$ Optimal basic feasible solution is

$(x_1, x_2, S_1, S_2, S_3, A_1, A_2) = (3, 6, 0, 0, 21, 0, 0)$

The optimal solution is unique as the C_j's for all non-basic variables ~~is be~~ are negative. We get an alternate solution only when the value of C_j for any non-basic variable is zero.

4.(a) Prove that the oscillation of a real-valued bounded function f defined on $[a,b]$ is the supremum of the set $\{|f(x_1)-f(x_2)| : x_1, x_2 \in [a,b]\}$

Sol^n: Let M, m be the bounds of 'f' on $[a,b]$.

Now $m \le f(x_1), f(x_2) \le M \quad \forall \; x_1, x_2 \in [a,b]$

$$|f(x_1)-f(x_2)| \le M-m \quad \forall \; x_1, x_2 \in [a,b] \quad \text{— (1)}$$

So $M-m$ is an upper bound of the set.

Let $\varepsilon > 0$ be any given number.

Since M is supremum of 'f', therefore $\exists \; x' \in [a,b]$ such that $f(x') > M - \frac{1}{2}\varepsilon$ — (2)

Similarly $\exists \; x'' \in [a,b]$ such that

$$f(x'') < m + \tfrac{1}{2}\varepsilon \quad \text{— (3)}$$

from equations (2) & (3) imply that $\exists \; x', x'' \in [a,b]$ such that $f(x') - f(x'') > M - m - \varepsilon$

$$|f(x') - f(x'')| > M - m - \varepsilon \quad \text{— (4)}$$

from (1) & (4) imply that $M-m$ is an upper bound and no number less than $M-m$ can be upper bound of the set in question.

So $M - m = \sup\{|f(x_1)-f(x_2)| : x_1, x_2 \in [a,b]\}$

4.(b) Classify the singular point $z=0$ of the function $f(z) = \frac{e^z}{z-\sin z}$ and obtain the principal part of its Laurent series expansion.

Sol^n Given $f(z) = \frac{e^z}{z-\sin z}$ and given that

$z=0$ is a singular point i.e $z-\sin z=0$ at $z=0$

So it is singular point, it is also called 'pole'.

Now let us write the expansion of e^z & $\sin z$ using taylor series so

$$f(z) = \frac{\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\right)}{\cancel{z}-\left(\cancel{z}-\frac{z^3}{6}+\frac{z^5}{120}-\cdots\right)}$$

$$= \frac{\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\right)}{\left(\frac{z^3}{6}-\frac{z^5}{120}+\frac{z^7}{7!}\cdots\right)}$$

$$= \frac{\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\right)}{\frac{z^3}{6}\left(1-\frac{z^2}{20}+\frac{z^4}{140}-\cdots\right)}$$

$$= \frac{6}{z^3}\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\right)\left(1-\left(\frac{z^2}{20}-\frac{z^4}{140}\cdots\right)\right)^{-1}$$

$$= \frac{6}{z^3}\left(1+z+\frac{z^2}{2}+\frac{z^3}{6}+\cdots\right)\left(1+\frac{z^2}{20}-\frac{z^4}{140}\cdots + \left(\frac{z^2}{20}-\frac{z^4}{140}\cdots\right)^2\cdots\right)$$

$$= \frac{6}{z^3}\left(1+z+\frac{z^2}{2}+\frac{z^3}{6}\cdots+\frac{z^2}{20}+\frac{z^3}{20}+\frac{z^4}{40}+\cdots -\frac{z^4}{140}-\frac{z^5}{140}\cdots+\frac{z^4}{400}+\frac{z^5}{400}+\cdots\right)$$

$$= \frac{6}{z^3}\left(1+z+\frac{22}{40}z^2+\frac{26}{120}z^3+z^4\left(\frac{1}{24}+\frac{1}{40}-\frac{1}{120}+\frac{1}{400}\right)+\cdots\right)$$

$$= \frac{6}{z^3}+\frac{6}{z^2}+\frac{66}{20z}+\frac{26\times 6}{120}+6\left(\frac{1}{24}+\frac{1}{40}-\frac{1}{120}+\frac{1}{400}\right)z+\cdots$$

So $f(z) = \frac{6}{z^3} + \frac{6}{z^2} + \frac{66}{20z} + \frac{26\times6}{120} +$

$z\times6\times\left(\frac{1}{24}+\frac{1}{40}-\frac{1}{120}+\frac{1}{40}\right) + \cdots)$

This is the laurent series expansion of $f(z)$.

we can clearly see that $z=0$ is a pole of order '3' This is the classification of singular point $z=0$.

And The principal part of $f(z)$ is

$\frac{6}{z^3} + \frac{6}{z^2} + \frac{66}{20z}$ Ans.

4. (c) A department head has 5 subordinates and 5 jobs to be performed. The time (in hours) that each subordinate will take to perform each job is given in the matrix below:

How should the jobs be assigned, one to each subordinate, so as to minimize the total time? Also, obtain the total minimum time to perform all the jobs if the subordinate IV cannot be assigned job C.

Jobs

Subordinates	A	B	C	D	E
I	4	9	4	12	4
II	15	11	20	5	8
III	17	7	15	12	18
IV	9	13	11	9	14
V	6	11	12	9	14

Solⁿ: Given a department head has 5 subordinates and 5 jobs to be performed with time in hour constraints. Also IV can't be assigned to job 'C'.

Aim to minimise time to finish jobs.

S.	A	B	C	D	E
I	4	9	4	12	4
II	15	11	20	5	8
III	17	7	15	12	18
IV	9	13	11	9	14
V	6	11	12	9	14

Let us solve by Hungarian Method

(Step 1) Substract min element or each row to all elements to that row.

	A	B	C	D	E
I	0	5	0	8	0
II	10	6	15	0	3
III	10	0	8	5	11
IV	0	4	2	0	5
V	0	5	6	3	8

Now step-2 Subtract min element of each row to all elements of that row.

The matrix after step 2 is same as in step-1 as all column has '0' as min element.

So

$$\begin{array}{c|ccccc} & A & B & C & D & E \\ \hline I & 0 & 5 & 0 & 8 & 0 \\ II & 10 & 6 & 15 & 0 & 3 \\ III & 10 & 0 & 8 & 5 & 11 \\ IV & 0 & 4 & 2 & 0 & 5 \\ V & 0 & 5 & 6 & 3 & 8 \end{array}$$

Step 3 Let's cover all zero's by min number horizontal & vertical lines.

no. of lines $r = 4$ no. of variables $n = 5$.

Clearly $r < n$ so optimality not achieved here.

So step 4: pick global min element not covered by lines & (i) substract it ato all uncovered elements

(ii) add it to all line junction elements

(iii) leave others as it is.

global min element is 2.

Then new matrix is

$$\begin{array}{c|ccccc} & A & B & C & D & E \\ \hline I & 2 & 5 & \cancel{0} & 10 & 0 \\ II & 10 & 4 & 13 & 0 & 1 \\ III & 12 & 0 & 8 & 7 & 11 \\ IV & \cancel{0} & 2 & 0 & \cancel{0} & 3 \\ V & 0 & 3 & 4 & 3 & 6 \end{array}$$

Step 4(i), (ii) (iii) done.

Now again following step 3

here no. of lines $r = 5 = n$ so the optimality has been achieved. Now Following the assignment of job $I \to E$, $II \to D$, $III \to B$, $IV \to C$, $V \to A$

So the cost is $\min = 4 + 5 + 7 + 11 + 6 = \underline{\underline{33}}$

Second part: Subordinate IV cannot be assigned to job C.

i.e, the problems indicates that one

assignment is impossible. Therefore, we put a very large cost (M) to the cell (4, 3) Then proceed as usual.
performing first row reduction and then column reduction, we get

	A	B	C	D	E
I	0	5	0	8	0
II	10	6	15	0	3
III	10	0	8	5	11
IV	0	4	M	0	5
V	0	5	6	3	8

Here cover all zeros by minimum number of horizontal and vertical lines.

no of lines $r = 4$, no. of variables $n = 5$
clearly $r < n$. so optimality is not reached.

Subtracting smallest element 3 from the uncovered elements and adding at intersection of covering lines namely 0 at (1,1), 8 at (1, 4), 10 at (3, 1) and 5 at (3, 4) and leave other covered elements unchanged.

The reduced cost-matrix so obtained is

	A	B	C	D	E
I	3	5	0	11	0
II	10	3	12	0	0
III	13	0	8	8	11
IV	0	1	M	0	2
V	0	2	3	3	5

Here no. of lines $r = 5 = n$.
So the optimality has been achieved.

For making assignments.

	A	B	C	D	E
I	3	5	0	11	$\cancel{0}$
II	10	3	12	$\cancel{0}$	0
III	13	0	8	8	11
IV	$\cancel{0}$	1	M	0	2
V	0	2	3	3	5

Each row and each column has one and only one assignment.
So an optimal assignment is reached.

$\therefore$ The optimal assignment is

$A \to V$, $B \to III$, $C \to I$

$D \to IV$, $E \to II$

The minimum assignment cost is

$C_{13} + C_{25} + C_{32} + C_{44} + C_{51}$

$= 4 + 8 + 7 + 9 + 6$

$= 34$

5.(a) Form a partial differential equation by eliminating arbitrary function f and g from
$$z = f(x^2-y) + g(x^2+y).$$

Solution:

Given $z = f(x^2-y) + g(x^2+y)$ —— (1)

Differentiating (1) partially w.r.t. x and y, we get,

$$\frac{\partial z}{\partial x} = 2x f'(x^2-y) + 2x g'(x^2+y) \quad \text{—— (2)}$$

and $$\frac{\partial z}{\partial y} = -f'(x^2-y) + g'(x^2+y) \quad \text{—— (3)}$$

Differentiating (2) partially w.r.t. x and (3) partially w.r.t. y, we get

$$\frac{\partial^2 z}{\partial x^2} = 2\{f'(x^2-y) + g'(x^2+y)\} + 4x^2\{f''(x^2-y) + g''(x^2+y)\} \quad \text{—— (4)}$$

and $$\frac{\partial^2 z}{\partial y^2} = f''(x^2-y) + g''(x^2+y) \quad \text{—— (5)}$$

From (2), we have $f'(x^2-y) + g'(x^2+y) = \frac{1}{2x}\frac{\partial z}{\partial x}$ —— (6)

Substituting from (5) and (6) in (4), we have,

$$\frac{\partial^2 z}{\partial x^2} = \cancel{x}\cdot\frac{1}{\cancel{x}x}\frac{\partial z}{\partial x} + 4x^2\cdot\frac{\partial^2 z}{\partial y^2}$$

$$\Rightarrow x\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial x} + 4x^3\frac{\partial^2 z}{\partial y^2}$$

$$\Rightarrow \boxed{xr = p + 4x^3 t}$$

which is the required partial differential equation of order 2.

Hence, the result.

5(b) Given $\frac{dy}{dx} = \frac{y^2-x}{y^2+x}$ with initial condition $y=1$ at $x=0$. find the value of y for $x=0.4$ by Euler's method, correct to 4 decimal places, taking step length $h=0.1$

Sol^n: Given $\frac{dy}{dx} = \frac{y^2-x}{y^2+x} = f(x,y)$

$y_0 = 1,\ x_0 = 0,\ y(0.4) = ?$

$h = 0.1$

By Euler's Method

$$y_1 = y_0 + hf(x_0, y_0)$$

$$y_1 = y(0.1) = 1 + (0.1)\left[\frac{y_0^2 - x_0}{y_0^2 + x_0}\right]$$

$$= 1 + 0.1\left[\frac{1-0}{1+0}\right]$$

$$= 1.1$$

$$y_2 = y(0.2) = y_1 + hf(x_1, y_1)$$

$$= 1.1 + hf(0.1, 1.1)$$

$$= 1.1 + (0.1)\left[\frac{(1.1)^2 - 0.1}{(1.1)^2 + 0.1}\right]$$

$$= 1.1 + 0.0847$$

$$= 1.1847$$

$$y_3 = y(0.3) = y_2 + hf(x_2, y_2)$$

$$= 1.1847 + (0.1)\left[\frac{(1.1847)^2 - 0.2}{(1.1847)^2 + 0.2}\right]$$

$$= 1.2585$$

$$y_4 = y(0.4) = y_3 + hf(x_3, y_3)$$

$$= 1.2585 + (0.1)\left[\frac{(1.2585)^2 - 0.3}{(1.2585)^2 + 0.3}\right]$$

$$= 1.3266$$

5(C) Evaluate, using the binary arithmetic, the following numbers in their given system:

(i) $(634.235)_8 - (132.223)_8$

(ii) $(7AB.4F32)_{16} - (5CA.D61)_{16}$

(i) Octal numbers can be converted into equivalent binary numbers by replacing each octal digit by its 3-bit equivalent binary.

$$(634.235)_8 = (110\ 011\ 100 . 010\ 011\ 101)_2$$

$$(132.223)_8 = (001\ 011\ 010 . 010\ 010\ 011)_2 \quad \text{— (1)}$$

$$\begin{array}{lr}
 & 110\ 011\ 100 . 010\ 011\ 101 \\
\text{1's complement of eqn (1)} \rightarrow & 110\ 100\ 101 . 101\ 101\ 100 \\
\hline
 & (1)\ 101\ 000\ 010 . 000\ 001\ 001 \\
 & 1 \quad \text{carry} \\
\hline
 & 101\ 000\ 010 . 000\ 001\ 010 \quad \text{Ans}
\end{array}$$

(ii) Hexadecimal numbers can be converted into equivalent binary numbers by replacing each hex digit by its equivalent 4-bit binary number.

$(7AB.432)_{16} = 0111\ 1010\ 1011 . 0100\ 0011\ 0010$

$(5CA.D61)_{16} = 0101\ 1100\ 1010 . 1101\ 0110\ 0001$ — (1)

(1) $0111\ 1010\ 1011 . 0100\ 0011\ 0010$

1's complement of eqn (1) → $1010\ 0011\ 0101 . 0010\ 1001\ 1110$

$0001\ 1110\ 0000 . 0110\ 1101\ 0000$ (1) carry

$0001\ 1110\ 0000 . 0110\ 1101\ 0001$ Ans

5(d) A planet of mass m is revolving around the Sun of mass M. The kinetic energy T and the potential energy V of the planet are given by

$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$ and $V = GMm\left(\frac{1}{2a} - \frac{1}{r}\right)$, where (r,θ) are the polar coordinates of the planet at time t, G is the gravitational constant and 2a is the major axis of the ellipse (the path of the planet). Find the Hamiltonian and the Hamilton equations of the planet's motion.

Solⁿ: Let r, θ be the instantaneous polar coordinates of a planet of mass m revolving around the Sun of mass M.

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \quad \text{--- (1)}.$$

$$V = GMm\left(\frac{1}{2a} - \frac{1}{r}\right) \quad \text{--- (2)}$$

where G is the gravitational constant and 2a is the major axis of the ellipse:

$$p_r = \frac{\partial T}{\partial \dot{r}} = m\dot{r}, \quad \dot{r} = \frac{p_r}{m} \quad \text{--- (3)}$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta}, \quad \dot{\theta} = \frac{p_\theta}{mr^2} \quad \text{--- (4)}$$

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + GMm\left(\frac{1}{2a} - \frac{1}{r}\right) \quad \text{--- (5)}$$

and the Hamiltonian equations are

$$\frac{\partial H}{\partial p_r} = \frac{p_r}{m} = \dot{r}, \quad \frac{\partial H}{\partial r} = -\frac{p_\theta^2}{mr^3} + \frac{GMm}{r^2} = -\dot{p}_r \quad \text{---} (6)$$

$$\frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}, \quad \frac{\partial H}{\partial \theta} = 0 = -\dot{p}_\theta \quad \text{---} (7)$$

Two equations in (7) show that

$$p_\theta = \text{Constant} = mr^2\dot{\theta} \quad \text{---} (8)$$

meaning the constancy of angular momentum or equivalently the constancy of areal velocity of the planet (Kepler's Second law of planetary motion).

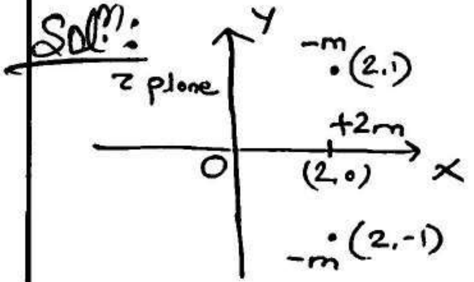
Two equations in (6) yield

$$\ddot{r} = \frac{\dot{p}_r}{m} = \frac{p_\theta^2}{m^2r^3} - \frac{GMm}{r^2} = r\dot{\theta}^2 - \frac{GMm}{r^2} \quad \text{---} (9)$$

Equation (9) describes the orbit of the planet (Kepler's first law of planetary motion).

5.(e) → In a fluid motion, there is a source of strength 2m placed at $z=2$ and two sinks of strength m are placed at $z=2+i$ and $z=2-i$. Find the streamlines.

Soln:



Complex potential w due to given arrangement at $P(z)$

$$= -2m\log(z-2) + m\log(z-(2+i)) + m\log(z-(2-i))$$

$$= m\log\{(z-2)^2+1\} - 2m\log(z-2)$$

$$w = m\log\{((x-2)+iy)^2+1\} - m\log\{(x-2)+iy\}^2$$

$$= m\log\{(x-2)^2-y^2+1+2iy(x-2)\}$$

$$- m\log\{(x-2)^2-y^2+2iy(x-2)\}$$

$\therefore w$ at $z=(x+iy)$

$$\phi+i\psi = m\log\{(x^2-y^2-4x+5)+2y(x-2)i\}$$

$$- m\log\{(x-2)^2-y^2+2iy(x-2)\}$$

$$\Rightarrow \psi = m\tan^{-1}\left(\frac{2y(x-2)}{x^2-y^2-4x+5}\right) - m\tan^{-1}\left(\frac{2y(x-2)}{x^2-y^2-4x+4}\right)$$

$$\Rightarrow \frac{\psi}{m} = \tan^{-1}\left\{\frac{2y(x-2)}{x^2-y^2-4x+5}\right\} - \tan^{-1}\left\{\frac{2y(x-2)}{x^2-y^2-4x+4}\right\}$$

$$= \tan^{-1}\left\{\frac{\dfrac{2y(x-2)(-1)}{(x^2-y^2-4x+5)(x^2-y^2-4x+4)}}{1+\dfrac{4y^2(x-2)^2}{(x^2-y^2-4x+5)(x^2-y^2-4x+4)}}\right\}$$

for eqn. of streamlines

$$\Rightarrow \frac{\psi}{m} = c \Rightarrow \tan\left(\frac{\psi}{m}\right) = c'$$

$$\Rightarrow \frac{-2y(x-2)}{(x^2-y^2-4x+5)(x^2-y^2-4x+4)+4y^2(x-2)^2} = c'$$

$$\Rightarrow \boxed{(x^2-y^2-4x+5)(x^2-y^2-4x+4)+4y^2(x-2)^2 = c''y(x-2)}$$

6(a) → Find the surface passing through the two lines $z=x=0$ and $z-1=x-y=0$, and satisfying the partial differential equation

$$\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = 0.$$

Solⁿ: The given differential equation may be written as

$$(D^2 - 4DD' + 4D'^2)z = 0$$

or $(D-2D')^2 z = 0.$

Its solution is $z = \phi_1(y+2x) + x\phi_2(y+2x)$ —— (1)

since (1) passes through $z=x=0$, we have

$0 = \phi_1(y)$ which gives $\phi_1(y+2x)=0.$

$\therefore$ (1) becomes $z = x\phi_2(y+2x).$ —— (2)

$\because$ (2) passes through $z-1=x-y=0$ i.e., $z=1$ and $y=x$,

$\therefore$ (2) gives $1 = x\phi_2(3x)$

or $\phi_2(3x) = 3/(3x)$

So that $\phi_2(y+2x) = 3/(y+2x).$

$\therefore$ from (2), $3x = z(y+2x)$

Which is the required surface.

6(b) Solve the system of linear equations

$7x_1 - x_2 + 2x_3 = 11$
$2x_1 + 8x_2 - x_3 = 9$
$x_1 - 2x_2 + 9x_3 = 7$

correct upto 4 significant figures by the Gauss-Seidel iterative method. Take initially guessed solution as

$x_1 = x_2 = x_3 = 0$

Solⁿ: Given system can be written as

$7x_1 - x_2 + 2x_3 = 11 \Rightarrow x_1 = \frac{1}{7}(11 + x_2 - 2x_3)$
$2x_1 + 8x_2 - x_3 = 9 \Rightarrow x_2 = \frac{1}{8}(9 - 2x_1 + x_3)$
$x_1 - 2x_2 + 9x_3 = 7 \Rightarrow x_3 = \frac{1}{9}(7 - x_1 + 2x_2)$

By Gauss-seidel method, above system can be written as

$$x_1^{(k+1)} = \frac{1}{7}\left(11 + x_2^{(k)} - 2x_3^{(k)}\right)$$
$$x_2^{(k+1)} = \frac{1}{8}\left(9 - 2x_1^{(k+1)} + x_3^{(k)}\right)$$
$$x_3^{(k+1)} = \frac{1}{9}\left(7 - x_1^{(k+1)} + 2x_2^{(k+1)}\right)$$

Now taking $x_1 = x_2 = x_3 = 0$, we obtain the following iterations.

$k=0$, $x_1^{(1)} = \frac{11}{7} = 1.5714$

$x_2^{(1)} = \frac{1}{8}(9 - 2(1.5714) + 0) = 0.7321$

$$x_3^{(1)} = \frac{1}{9}\left(7 - 1.5714 + 2(0.7321)\right) = 0.7659$$

<u>k=1</u>;

$$x_1^{(2)} = \frac{1}{7}\left(11 + 0.7321 - 2(0.7659)\right) = 1.4572$$

$$x_2^{(2)} = \frac{1}{8}\left(9 - 2(1.4572) + 0.7659\right) = 0.8564$$

$$x_3^{(2)} = \frac{1}{9}\left(7 - 1.4572 + 2(0.8564)\right) = 0.8062$$

<u>k=2</u>; $x_1^{(3)} = 1.4634$, $x_2^{(3)} = 0.8599$, $x_3^{(3)} = 0.8062$

<u>k=3</u>: $x_1^{(4)} = 1.4639$, $x_2^{(4)} = 0.8598$, $x_3^{(4)} = 0.8062$

<u>k=4</u>: $x_1^{(5)} = 1.4639$, $x_2^{(5)} = 0.8598$, $x_3^{(5)} = 0.8062$

$\therefore$ $x_1 = 1.4639$

$x_2 = 0.8598$

$x_3 = 0.8062$

which is the required solution correct to 4 significant figures.

6.(c) → A mechanical system with 2 degrees of freedom has the Lagrangian

$$L=\frac{1}{2}m(\dot{x}^2+\dot{y}^2)-\frac{1}{2}m(\omega_1^2x^2+\omega_2^2y^2)+Kxy$$

where m, ω_1, ω_2, K are constants. Find the parameter θ so that under the transformation

$$x=q_1\cos\theta-q_2\sin\theta,\quad y=q_1\sin\theta+q_2\cos\theta$$

the Lagrangian in terms of q_1, q_2 will not contain the product term q_1q_2. Find the Lagrange's equations w.r.t. q_1 and q_2 independent of parameter θ.

Solⁿ: Lagrange's equation $L=T-V$

When co-ordinate system is x-y (2D cartesian)

$$T=\frac{m}{2}(\dot{x}^2+\dot{y}^2)$$

$$\therefore V=T-L=\frac{1}{2}m(\omega_1^2x^2+\omega_2^2y^2)-Kxy$$

Under transformation

$$x=q_1\cos\theta-q_2\sin\theta,\quad y=q_1\sin\theta+q_2\cos\theta$$

$$\dot{x}=-q_1\sin\theta-q_2\cos\theta$$

$$\dot{y}=q_1\cos\theta-q_2\sin\theta$$

$$T=\frac{m}{2}(\dot{q}_1^2+\dot{q}_2^2)$$

$$V=\frac{m}{2}\left\{\omega_1^2(q_1\cos\theta-q_2\sin\theta)^2+\omega_2^2(q_1\sin\theta+q_2\cos\theta)^2\right\}$$
$$-K(q_1\cos\theta-q_2\sin\theta)(q_1\sin\theta+q_2\cos\theta)$$

$$=\frac{m}{2}\left\{q_1^2(\omega_1^2\cos^2\theta+\omega_2^2\sin^2\theta)+q_2^2(\omega_1^2\sin^2\theta+\omega_2^2\cos^2\theta)\right.$$
$$\left.+2q_1q_2\sin\theta\cos\theta(\omega_2^2-\omega_1^2)\right\}$$
$$-K\left(q_1q_2(\cos^2\theta-\sin^2\theta)+\sin\theta\cos\theta(q_1^2-q_2^2)\right)$$

$$= \frac{m}{2}\left\{ q_1^2\left(\frac{\omega_1^2+\omega_2^2}{2} + \frac{\cos 2\theta}{2}(\omega_1^2-\omega_2^2)\right) + \right.$$
$$q_2^2\left(\frac{\omega_1^2+\omega_2^2}{2} + \frac{\cos 2\theta}{2}(\omega_2^2-\omega_1^2)\right) +$$
$$\left. 2q_1q_2 \sin\theta\cos\theta(\omega_2^2-\omega_1^2)\right\} -$$
$$K\left(q_1q_2(\cos^2\theta-\sin^2\theta) + \sin\theta\cos\theta(q_1^2-q_2^2)\right)$$

$$= \frac{m}{2}\left\{(q_1^2+q_2^2)\left(\frac{\omega_1^2+\omega_2^2}{2}\right) + (q_1^2-q_2^2)(\omega_1^2-\omega_2^2)\left(\frac{\cos 2\theta}{2}\right)\right.$$
$$\left. + 2q_1q_2\sin\theta\cos\theta(\omega_2^2-\omega_1^2)\right\}$$
$$-K\left(q_1q_2(\cos^2\theta-\sin^2\theta) + \sin\theta\cos\theta(q_1^2-q_2^2)\right)$$

For q_1q_2 term to vanish

$$0 = \frac{m}{2}\left\{2\sin\theta\cos\theta(\omega_2^2-\omega_1^2)\right\} - K\left((\cos^2\theta-\sin^2\theta)\right)$$
$$= \frac{m}{2}\left\{(\omega_2^2-\omega_1^2)\sin 2\theta\right\} - K\cos 2\theta$$

i.e., $$\boxed{\theta = \frac{1}{2}\tan^{-1}\frac{2K}{m(\omega_2^2-\omega_1^2)}}$$

$$\sin 2\theta = \frac{2K}{\sqrt{4K^2+m^2(\omega_2^2-\omega_1^2)}} = A$$

$$\cos 2\theta = \frac{m(\omega_2^2-\omega_1^2)}{\sqrt{4K^2+m^2(\omega_2^2-\omega_1^2)}} = B$$

$$\boxed{L = \frac{m}{2}(\dot{q}_1^2+\dot{q}_2^2) - \frac{m}{2}\left\{(q_1^2+q_2^2)\left(\frac{\omega_1^2+\omega_2^2}{2}\right) + (q_1^2-q_2^2)(\omega_1^2-\omega_2^2)\left(\frac{B}{2}\right)\right\} + K\left(\frac{A}{2}(q_1^2-q_2^2)\right)}$$

7.(a) (i) Find the conjunctive normal form (CNF) of the following Boolean function: $f(x,y,z,t) = xyz + y\bar{x}(t+\bar{z})$

(ii) Express the Boolean function $f(x,y,z) = x + \overline{(\bar{x}\bar{y} + \bar{x}\,z)} + z$ in disjunctive normal form (DNF) and construct the truth table for the function.

Soln (i) Given Boolean function is

$$f(x,y,z,t) = xyz + y\bar{x}(t+\bar{z})$$
$$= xyz + \bar{x}yt + \bar{x}y\bar{z} \quad \left(\text{By distributive law}\right).$$

A conjunctive normal form of boolean expression is form of product of sums i.e. it will have maxterms of 'n' literals in product form.

The given expression is in sum of products form. So let us convert it in product of sums i.e. CNF.

Let us write 'f' in DNF then we will apply De Morgan's law to convert $f \to f'$ which is CNF.

$$\text{So } f(x,y,z,t) = xyz(t+\bar{t}) + \bar{x}yt(z+\bar{z}) + \bar{x}y\bar{z}(t+\bar{t})$$

$\because x+\bar{x} = 1$ & anding 1 does not alter.

$$= xyzt + xyz\bar{t} + \bar{x}ytz + \bar{x}yt\bar{z} + \bar{x}y\bar{z}t + \bar{x}y\bar{z}\bar{t}$$

which is in DNF form now i.e. SOP form.

Using De' Morgan's law $\bar{f}$ will be in CNF.

So
$$\overline{f(x,y,z,t)} = (\bar{x}+\bar{y}+\bar{z}+\bar{t})\cdot(\bar{x}+\bar{y}+\bar{z}+t)\cdot(x+\bar{y}+\bar{t}+\bar{z})\cdot(x+\bar{y}+\bar{t}+z)\cdot(x+\bar{y}+z+\bar{t})\cdot(x+\bar{y}+z+t)$$

Clearly $F(x,y,z,t) = \overline{f(x,y,z,t)}$ is in CNF.

(ii) Given $f(x,y,z) = x + (\overline{\bar{x}\cdot\bar{y} + \bar{x}\cdot z}) + z$

Need to convert 'f' in Disjunctive Normal form (DNF) i.e in form of Sum of Products. i.e minterms usages in sum form.

$\because \left(\overline{\bar{x}\cdot\bar{y} + \bar{x}\cdot z}\right) = \left(\overline{\bar{x}\cdot\bar{y}}\right)\cdot\left(\overline{\bar{x}\cdot z}\right)$ using De morgan's Law.

$= (x+y)\cdot(x+\bar{z})$.

So $f(x,y,z) = x + (x+y)\cdot(x+\bar{z}) + z$

$= x + x\cdot x + y\cdot x + x\bar{z} + y\bar{z} + z$

$= x(1+y) + x(\bar{z}+z) + y\bar{z}$ $\quad \boxed{\because x\cdot x = x,\ x+x = x}$

$= x + x + y\bar{z}$ $\quad \boxed{1+y=1}$

$= x + y\bar{z}$ $\quad \boxed{\bar{z}+z=1}$

Now the resolved 'f' can be written SOP form by adding term's not present.

$\boxed{\because y+\bar{y}=1,\ 1\cdot x = x}$

$f(x,y,z) = x(y+\bar{y}) + y\bar{z}(x+\bar{x})$

$= xy + x\bar{y} + xy\bar{z} + \bar{x}y\bar{z}$

$= xyz + \underline{xy\bar{z}} + x\bar{y}z + x\bar{y}\bar{z} + \underline{xy\bar{z}} + \bar{x}y\bar{z}$

$= xyz + xy\bar{z} + x\bar{y}z + \bar{x}y\bar{z} + x\bar{y}\bar{z}$

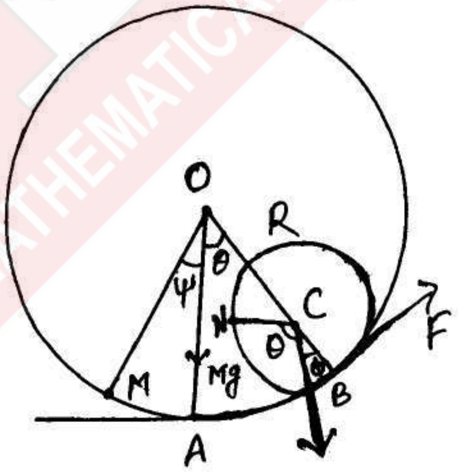
is required DNF (SOP) form of 'f'.

Now truth table

x	y	z		f	
0	0	0	$\longrightarrow$	0	$\longrightarrow 0+0+0+0+0$
0	0	1	$\longrightarrow$	0	$\longrightarrow 0+0+0+0$
0	1	0	$\longrightarrow$	1	$\longrightarrow 0+0+0+1+0$
0	1	1	$\longrightarrow$	0	$\longrightarrow$
1	0	0	$\longrightarrow$	1	$\longrightarrow 0+0+0+0+1$
1	0	1	$\longrightarrow$	1	
1	1	0	$\longrightarrow$	1	
1	1	1	$\longrightarrow$	1	

7.(b) → A perfectly rough ball is at rest within a hollow cylindrical roller. The roller is drawn along a level path with uniform velocity V. Let a and b be the raddi of the ball and the roller respectively. If $V^2 > \frac{27}{7} g(b-a)$, then show that the ball will roll completely round the inside of the roller.

Solⁿ: Let O be the centre of the roller and C the centre of the spherical ball moving inside the cylindrical roller. Let CN be the radius of the ball which was vertical when it was in its lowest position. When the roller has moved through a distance x, let CN have turned through an angle θ. The line joining the centre makes an angle ϕ with the vertical and the ball has turned through an angle θ. As there is no sliding,



arc BM = arc BN

i.e. $b(\phi+\psi) = a(\theta+\phi)$ ——— (1)

or $(b-a)\phi = a\theta - b\psi$

Again the velocity of the roller is constant

i.e. $\dot{x} = b\dot{\psi} = V$.

Then $\ddot{x} = b\ddot{\psi} = 0$. ——— (2)

Let R and F be the normal reaction and friction.

As C describes a circle of radius $(b-a)$ about O, so accelerations along CO and perpendicular to CO are $(b-a)\dot{\phi}^2$ and $(b-a)\ddot{\phi}$ respectively.

Thus equation of motion are

$$m(b-a)\dot{\phi}^2 = R - mg\cos\phi, \quad \text{——— (3)}$$

$$m(b-a)\ddot{\phi} = F - mg\sin\phi, \quad \text{——— (4)}$$

and $$m\frac{2a^2}{5}\ddot{\theta} = -F.a, \quad \text{——— (5)}$$

Eliminating F between (4) and (5), we get

$$(b-a)\ddot{\phi} = -\frac{2a}{5}\ddot{\theta} - g\sin\phi$$

or $$(b-a)\ddot{\phi} + \frac{2}{5}(b-a)\ddot{\phi} = -g\sin\phi$$

$$\left[\because (b-a)\ddot{\phi} = a\ddot{\theta} - a\ddot{\psi} = a\ddot{\theta} \text{ by virtue of (2)}\right]$$

or $$\frac{7}{5}(b-a)\ddot{\phi} = -g\sin\phi. \quad \text{——— (6)}$$

Integrating it, we get

$$\frac{7}{5}(b-a)\dot{\phi}^2 = 2g\cos\phi + A. \quad \text{——— (7)}$$

Initially the velocity of the C.G. is

$$\dot{x} + (b-a)\dot{\phi} = 0,$$

i.e. $$(b-a)\dot{\phi} = -\dot{x} = -V.$$

$$\therefore A = \frac{7V^2}{5(b-a)} - 2g.$$

Hence the equation (7) gives

$$\frac{7}{5}(b-a)\dot{\phi}^2 = -2g(1-\cos\phi) + \frac{7V^2}{5(b-a)}. \quad \text{——— (8)}$$

Substituting for $\dot{\phi}^2$ from ⑧ in ③, we get

$$\frac{R}{m} = g\cos\phi + \frac{V^2}{b-a} - \frac{10}{7}(-\cos\phi)$$

$$= \frac{1}{7}\left(17g\cos\phi - 10g + \frac{7V^2}{b-a}\right)$$

The necessary condition that the ball should roll completely round the fixed cylinder is that R is positive when $\phi = \pi$, and if R is positive in this position, when it will be positive in all positions.

Hence $\left\{\frac{7V^2}{b-a} - 10 + 17g\cos\phi\right\}_{\phi=\pi} > 0$

or $\frac{7V^2}{b-a} > 27g$

or $V^2 > \frac{27g(b-a)}{7}$.

7.(c) Solve the partial differential equation

$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$ $0 < x < L$, $t > 0$ subject to conditions

$u(0,t) = 0$, $u(L,t) = 0$ $t > 0$

$u(x,0) = x$, $\left.\frac{\partial u}{\partial t}\right)_{t=0} = 1$ $0 < x < L$.

Sol^n: The given partial differential Equation is wave equation, rewritten as $\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}$

$0 < x < L$, $t > 0$.

The given conditions $u(0,t) = 0$, $u(L,t) = 0$ $t > 0$

so let the solution of this PDE is $u = X(x)T(t)$

Then $\frac{\partial u}{\partial x} = X'T$ $\frac{\partial^2 u}{\partial x^2} = X''T$. $\frac{\partial u}{\partial t} = XT'$

$\frac{\partial^2 u}{\partial t^2} = XT''$. Now putting these in wave eq^n

we get $X''T = \frac{1}{a^2}XT''$ $\because$ in wave equation the independent variables x, t are independent of each other so we can make the PDE as separate

$\frac{X''}{X} = \mu$, $\frac{T''}{aT} = \mu$ Now we can solve them to get G.S.

Initial condition. $u(0,t) = 0 \Rightarrow X(0)T(t) = 0$

$\because T(t)$ is non zero function $\Rightarrow X(0) = 0$ —(i)

$u(L,t) = 0 \Rightarrow X(L)T(t) = 0 \Rightarrow X(L) = 0$ —(ii).

$u(x,0) = x \Rightarrow X(x)T(0) = x$ & $\left.\frac{\partial u}{\partial t}\right|_{t=0} = 1$

So let us solve the PDE's separately as $\frac{X''}{X} = \mu$

or $X'' - \mu X = 0$

Case I: let $\mu = 0$ then $X'' = 0 \Rightarrow X(x) = A(x) + B$

$\because X(0) = 0 \Rightarrow 0 = A\cdot 0 + B \Rightarrow \boxed{B = 0}$

and $X(L) = 0 \Rightarrow 0 = A\cdot L + 0 \Rightarrow \boxed{A = 0}$ i.e $X(x) = 0$ which can't be the case. So Rejecting it.

Case II : let $\mu = \lambda^2$ $\lambda > 0$ then $X'' - \lambda^2 X = 0$
The general solution of it is $m^2 - \lambda^2 = 0$ $m = \pm\lambda$
$X(x) = Ae^{\lambda x} + Be^{-\lambda x}$. Now $X(0) = 0$ & $X(L) = 0$
when used we again get $A=0, B=0$ as $e^{\lambda x}, e^{-\lambda x}$ can't be zero. So again rejecting this case.

Case III : let $\mu = -\lambda^2$ $\lambda > 0$, then $X'' + \lambda^2 X = 0$
The general solution of this is $m^2 + \lambda^2 = 0$ $m = \pm\lambda i$
or $X(x) = A\cos\lambda x + B\sin\lambda x$.
Now $X(0) = 0 \Rightarrow 0 = A + 0 \Rightarrow A = 0$
Now $X(L) = 0 \Rightarrow 0 = B\sin\lambda L$ clearly B can't be zero $X(x)$ is not zero. so $\sin\lambda L = 0$
or $\lambda L = n\pi \Rightarrow \lambda = \frac{n\pi}{L}$ for $n = 1, 2, \ldots$

So we get $X(x) = \sum_n B_n \sin\left(\frac{n\pi}{L}x\right)$

Now 2nd PDE. $\frac{T''}{a^2 T} = \mu$ will be solved by taking $\mu = -\lambda^2$ as above or $T'' + a^2\lambda^2 T = 0$ or
or $m^2 + a^2\lambda^2 = 0 \Rightarrow m = \pm a\lambda i$
$T(t) = C\cos a\lambda t + D\sin a\lambda t$

So $u = \sum u_n = \sum_{n=1}^{\infty}\left(C_n\cos\left(\frac{an\pi}{L}t\right) + D_n\sin\left(\frac{an\pi}{L}t\right)\right) \cdot B_n\sin\left(\frac{n\pi}{L}x\right)$

$\because u(x,0) = x \Rightarrow x = \sum_{n=1}^{\infty}\left(C_n\cos\left(\frac{an\pi}{L}t\right) + D_n \cdot 0\right) B_n\sin\left(\frac{n\pi}{L}x\right)$

And $\left.\frac{\partial u}{\partial t}\right|_{t=0} = 1 \Rightarrow 1 = \left(-C_n\frac{an\pi}{L}\sin\left(\frac{an\pi}{L}t\right) + D_n\frac{an\pi}{L}\cos\left(\frac{an\pi}{L}t\right)\right) B_n \sin\frac{n\pi x}{L}$

$1 = 0 + D_n\frac{an\pi}{L} \cdot 1 \cdot B_n\sin\frac{n\pi x}{L}$

$1 = D_n \cdot B_n \frac{an\pi}{L}\sin\frac{n\pi x}{L}$

OR $u = \sum_{n=1}^{\infty}\left(E_n \cos\left(\frac{an\pi}{L}t\right) + F_n \sin\left(\frac{an\pi}{L}t\right)\right)\sin\left(\frac{n\pi}{L}x\right)$

where $E_n = B_n C_n$ $F_n = B_n D_n$. arb const.

and $1 = \sum_{n=1}^{\infty} F_n \frac{an\pi}{L} \sin\frac{n\pi}{L}x$ —— & ⓐ

$x = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi}{L}x$ —— ⓑ

from ⓑ & ⓐ which is fourier sine series representation of 'x' & '1' we can get

$E_n = \frac{2}{L}\int_0^L x \sin\left(\frac{n\pi}{L}x\right) dx$ —— ⓒ $\boxed{L = l}$

& $\frac{an\pi}{L} F_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi}{L}x\right) dx$ —— ⓓ

Then the solution

$u(x,t) = \sum_{n=1}^{\infty}\left(E_n \cos\left(\frac{n\pi a}{L}t\right) + F_n \sin\left(\frac{an\pi}{L}t\right)\right)\sin\left(\frac{n\pi}{L}x\right)$

from ⓓ: $\frac{an\pi}{\cancel{L}} F_n = \frac{2}{\cancel{L}}\left[-\cos\frac{n\pi}{L}x\right]_0^L \cdot \frac{L}{n\pi}$

$an\pi F_n = \frac{2}{n\pi}L\left[-\cos n\pi + 1\right]$

so $F_n = \frac{2L}{an^2\pi^2}[2]$ for odd values of n

& $F_n = 0$ for even 'n'.

from ⓒ: $E_n = \frac{2}{l}\left[\left[x\frac{L}{n\pi}-\cos\left(\frac{n\pi}{L}x\right)\right]_0^l - \int_0^l \frac{L}{n\pi}\left(-\cos\frac{n\pi}{L}x\right)dx\right]$

$= \frac{2}{l}\left[\left(\frac{-L^2}{n\pi}\cos(n\pi)\right) + \frac{L^2}{n^2\pi^2}\left[\sin\frac{n\pi}{L}x\right]_0^l\right]$

0 for both 'n' even odd.

$E_n = \frac{2}{l}\left[-\frac{L^2}{n^2\pi^2}\cos(n\pi)\right] = \frac{2L}{n^2\pi^2}\begin{cases}-1 & \text{for } n \text{ even.}\\ +1 & \text{for } n \text{ odd.}\end{cases}$

or $u(x,t)$ in case n is odd

$$= \sum_{n=1}^{\infty} \frac{2l}{n^2\pi^2} \cos\left(\frac{n\pi a}{L}t\right) \sin\left(\frac{n\pi}{L}x\right)$$

$u(x,t)$ in case n is even.

$$= \sum_{n=1}^{\infty} \left[\frac{-2l}{n^2\pi^2} \cos\left(\frac{n\pi a}{L}t\right) + \frac{4L}{n^2\pi^2 a} \sin\left(\frac{n\pi a}{L}t\right)\right] \sin\left(\frac{n\pi}{L}x\right).$$

Ans.

8.(a) → Reduce the partial differential equation

$$\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\left(1+\frac{1}{x}\right) + \frac{z}{x} = 0$$

to canonical form.

Solⁿ: Given

$$0.r - s + t + p - q\left(1+\frac{1}{x}\right) + (z/x) = 0 \quad \text{——— (1)}$$

Comparing (1) with $Rr + Ss + Tt + f(x,y,z,p,q) = 0$,

here $R=0$, $S=-1$ and $T=1$.

Hence $S^2 - 4RT = 1 > 0$, showing that the given equation is hyperbolic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $-\lambda + 1 = 0$ giving $\lambda = 1$.

Hence the corresponding characteristic equation $dy/dx + \lambda = 0$ yields $dy/dx + 1 = 0$ or $dx + dy = 0$

Integrating it, $x + y = c$, c being an arbitrary constant.

Choose $u = x+y$ and $v = x$, ——— (2)

where we have chosen $v = x$ in such a manner that u and v are independent as verified below:

$$\text{Jacobian of } u \text{ and } v = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1 \neq 0$$

$\Rightarrow$ u and v are independent functions.

Now, $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$, using (2) ——— (3)

$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u}$, using (2) ——— (4)

from (4), we have $\partial/\partial y \equiv \partial/\partial u$ ——— (5)

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \quad \text{——— (6)}$$

using (3) and (5)

and $t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)$, using (5)

or $t = \partial^2 z/\partial u^2$. ——— (7)

using (2), (3), (4), (6) and (7), (1) reduces to

$$-\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v}\right) + \frac{\partial^2 z}{\partial u^2} + \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\left(1 + \frac{1}{v}\right) + \frac{z}{v} = 0$$

or $\partial^2 z/\partial u \partial v - (\partial z/\partial v) + (1/v) \times (\partial z/\partial u) - (z/v) = 0$,

which is the required canonical form.

8.(b) → Compute a root of the equation $\log_{10}(2x+1)-x^2+3=0$, in the interval $[0,3]$, by Regula-Falsi method, correct to 6 decimal places.

Soln: Let $f(x)=\log_{10}(2x+1)-x^2+3$

so that $f(0)=3$ and $f(3)=\log_{10}7-9+3=-5.1549$

i.e. the root lies between 0 and 3.

Taking $x_0=0$, $x_1=3$, $f(x_0)=3$ and $f(x_1)=-5.1549020$ in the regula-falsi method we get

$$x_2=x_0-\frac{x_1-x_0}{f(x_1)-f(x_0)}f(x_0)=0-\frac{3-0}{-5.1549020-3}$$

$$x_2=0.3678769$$

$$f(x_2)=3.1041547$$

i.e. the root lies between x_2 and x_1.

Taking $x_0=0.3678769$, $x_1=3$

$f(x_0)=3.1041547$, $f(x_1)=-5.1549020$

$$x_3=x_0-\frac{x_1-x_0}{f(x_1)-f(x_0)}f(x_0)$$

$$x_3=\frac{x_0f(x_1)-x_1f(x_0)}{f(x_1)-f(x_0)}f(x_0)=1.3571566$$

$$f(x_3)=1.7280045$$

The root lies between x_3 and x_1 and x_3 treated as x_0.

Similarly,

$$x_4=x_0-\frac{x_1-x_0}{f(x_1)-f(x_0)}f(x_0)=1.7696046$$

$$f(x_4)=0.5254798.$$

The root lie between x_4 and x_1 and x_4 treated as x_0.

$$x_5 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 1.8834258$$

$f(x_5) = 0.1309389$

The root lie between x_5 and x_1 and x_5 treated as x_0.

$$x_6 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 1.9110852$$

$f(x_6) = 0.0309959$

The root lie between x_6 and x_1 and x_6 treated as x_0.

$$x_7 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 1.9175936$$

$f(x_7) = 0.00724831$

$\therefore$ The root is $x = 1.917594$.

8.(c) → Determine under what conditions the velocity field $u = c(x^2 - y^2)$, $v = -2cxy$, $w = 0$ is a solution to the Navier-stokes momentum equations. Assuming that the conditions are met, determine the resulting pressure distribution, when z is up and the external body forces are $B_x = 0 = B_y$, $B_z = -g$.

Solⁿ: velocity field is $u = c(x^2 - y^2)$, $v = -2cxy$, $w = 0$

To determine the given velocity field is a solution of the Navier-stokes momentum equation, it should satisfy the continuity equation that is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

so $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(c(x^2 - y^2)\right) = 2cx$,

$$\frac{\partial^2 u}{\partial x^2} = 2c, \quad \frac{\partial^2 u}{\partial y^2} = -2c$$

and $\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(-2cxy) = -2cx$, $\frac{\partial^2 v}{\partial y^2} = 0$,

$$\frac{\partial^2 v}{\partial y \partial x} = -2c, \quad \frac{\partial^2 v}{\partial x^2} = 0 \text{ and } \frac{\partial w}{\partial z} = 0$$

So, L.H.S. $= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 2cx - 2cx + 0 = 0 =$ R.H.S.

Hence, it is the solution to the Navier-stokes momentum equation.

Now we find the resulting pressure distribution, that is $p(x, y, z, t)$. We have $B_x = 0 = B_y$, $B_z = -g$.

Since $\frac{\partial u}{\partial t} = 0$, $\frac{\partial v}{\partial t} = 0$, $\frac{\partial w}{\partial t} = 0$. So, the velocity variables are independent of time and the flow is steady.

Now since the flow is steady, so we find the $p(x,y,z)$.

Now using the Navier-Stokes momentum equation,

In x-direction

$$\rho B_x - \frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] = \rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right]$$

$$\Rightarrow -\frac{\partial p}{\partial x} = \rho\left[2c^2x^3 + 2c^2xy^2\right] = 2c^2\rho x(x^2+y^2) \quad \text{——} (1)$$

Now in y-direction

$$\rho B_y - \frac{\partial p}{\partial y} + \mu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right] = \rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right]$$

$$\Rightarrow -\frac{\partial p}{\partial y} = \rho\left[-2c^2x^2y + 2c^2y^3 + 4c^2x^2y\right] = 2c^2\rho y(x^2+y^2) \quad \text{——} (2)$$

Now in z-direction

$$\frac{\partial p}{\partial z} = -\rho B \quad \text{——} (3)$$

$$\text{eq}^n (1) \Rightarrow p = \int \frac{\partial p}{\partial x}dx = \int -2c^2\rho x(x^2+y^2)\,dx$$

$$= \int -2c^2\rho(x^3+xy^2)\,dx = -2c^2\rho\left[\frac{x^4}{4} + \frac{x^2y^2}{2}\right] + f_1(y,z)$$

Therefore $p = -2c^2\rho\left[\frac{x^4}{4} + \frac{x^2y^2}{2}\right] + f_1(y,z)$ —— (4)

where $f_1(y,z)$ is a constant of functions y and z.

Now derivate the above equation w.r.t. to y, then

$$\frac{\partial p}{\partial y} = \frac{\partial}{\partial y}\left(-2c^2\rho\left[\frac{x^4}{4} + \frac{x^2y^2}{2}\right] + f_1(y,z)\right) = -2c^2\rho x^2y + \frac{\partial f_1}{\partial y} \quad \text{——} (5)$$

Comparing equations (2) and (5) implies

$$\frac{\partial f_1}{\partial y} = \int \frac{\partial f_1}{\partial y}dy = \int -2c^2\rho y^3 dy = -2c^2\rho\frac{y^4}{4} + f_2(z)$$

where $f_2(z)$ is a constant of function z.

Putting the values of $f_1(y,z)$ in equation (4),

$$P = -2c^2\rho\left[\frac{x^4}{4}+\frac{x^2y^2}{2}\right]+f_1(y,z) = -2c^2\rho\left[\frac{x^4}{4}+\frac{x^2y^2}{2}\right]-2c^2\rho\frac{y^4}{4}+f_2(z) \quad \text{—— (6)}$$

derivate the above equation w.r.t. to z. then

$$\frac{\partial P}{\partial z} = \frac{\partial}{\partial z}\left(-2c^2\rho\left[\frac{x^4}{4}+\frac{x^2y^2}{2}\right]-2c^2\rho\frac{y^4}{4}+f_2(z)\right) = 0+\frac{\partial f_2}{\partial z} \quad \text{—— (7)}$$

compare equation (3) and (7), then

$$\frac{\partial P}{\partial z} = -\rho B = \frac{\partial f_2}{\partial z} \Rightarrow \frac{\partial f_2}{\partial z} = -\rho B$$

Integrate this w.r.t. to z, implying

$$f_2(z) = \int -\rho B\,dz = -\rho Bz + C,$$

where c is a constant.

Putting the value of $f_2(z)$ in eq. (7),

$$P = -2c^2\rho\left[\frac{x^4}{4}+\frac{x^2y^2}{2}\right]-2c^2\rho\frac{y^4}{4}+f_2(z) = -2c^2\rho\left[\frac{x^4}{4}+\frac{x^2y^2}{2}\right]-2c^2\rho\frac{y^4}{4}-\rho Bz+C.$$

Thus, $P = -2c^2\rho\left[\frac{x^4}{4}+\frac{x^2y^2}{2}+\frac{y^4}{4}\right]-\rho Bz+C$

The resulting presure distribution is

$$P(x,y,z) = -2c^2\rho\left[\frac{x^4}{4}+\frac{x^2y^2}{2}+\frac{y^4}{4}\right]-\rho Bz+C$$

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