IMS™

(INSTITUTE OF MATHEMATICAL SCIENCES)

UPSC-CSE 2023

Mains

MATHEMATICS

Optional Paper-I

Solutions

1(a) Let $v_1 = (2,-1,3,2)$, $v_2 = (-1,1,1,-3)$ and $v_3 = (1,1,9,-5)$ be three vectors of the space $\mathbb{R}^4$. Does $(3,-1,0,-1) \in \text{span}\{v_1, v_2, v_3\}$? Justify your answer.

Sol. Let $(3,-1,0,-1) = x(2,-1,3,2) + y(-1,1,1,-3) + z(1,1,9,-5)$, $x,y,z \in \mathbb{R}$. ——— (1)

$$\Rightarrow \left.\begin{aligned} 2x-y+z &= 3 \\ -x+y+z &= -1 \\ 3x+y+9z &= 0 \\ 2x-3y-5z &= 1 \end{aligned}\right\} \text{——— (2)}$$

$\Longrightarrow AX = B$ ——— (3)

where $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 1 \\ 3 & 1 & 9 \\ 2 & -3 & -5 \end{bmatrix}$ $\begin{bmatrix} 3 \\ -1 \\ 0 \\ 1 \end{bmatrix} = B$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

We have $[A|B] = \left[\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ -1 & 1 & 1 & -1 \\ 3 & 1 & 9 & 0 \\ 2 & -3 & -5 & 1 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} -1 & 1 & 1 & -1 \\ 2 & -1 & 1 & 3 \\ 3 & 1 & 9 & 0 \\ 2 & -3 & -5 & 1 \end{array}\right]$ $R_1 \leftrightarrow R_2$

$$[A|B] \sim \left[\begin{array}{ccc|c} -1 & 1 & 1 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 4 & 12 & -3 \\ 0 & -1 & -3 & -1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 + 3R_1 \\ R_4 \to R_4 + 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} -1 & 1 & 1 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 8 & 1 \\ 0 & 0 & -2 & -2 \end{array}\right] \quad \begin{array}{l} R_3 \to R_4 - 4R_1 \\ R_4 \to R_4 + R_1 \end{array}$$

$$R_4 \to 4R_4$$

$$\sim \left[\begin{array}{ccc|c} -1 & 1 & 1 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 8 & 1 \\ 0 & 0 & -8 & -8 \end{array}\right]$$

$$R_4 \to R_4 + R_3$$

$$\sim \left[\begin{array}{ccc|c} -1 & 1 & 1 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 8 & 1 \\ 0 & 0 & 0 & -7 \end{array}\right]$$

clearly it is in echelon form

$\therefore\ \rho(A) = 3,\quad \rho(A|B) = 4,$

$\therefore\ \rho(A) \neq \rho(A|B),$

$\therefore$ The given system of equations have no solution for x, y, z.

$\therefore\ (3,-1,0,1)$ cannot be expressed as linear combination of v_1, v_2, v_3.

$\therefore\ (3,-1,0,1) \notin \text{span}\{v_1, v_2, v_3\}$.

(OR)

1.(a) Given $V_1 = (2,-1,3,2)$ $V_2 = (-1,1,1,-3)$
$V_3 = (1,1,9,-5)$ be three vectors of space R^4.
To check if $V_4 = (3,-1,0,1)$ is spanned by $\{V_1, V_2, V_3\}$
i.e is V_4 a linear combination of $\{V_1, V_2, V_3\}$.
So let $x, y, z \in R$ such that

i.e $V_4 = xV_1 + yV_2 + zV_3$

$(3,-1,0,1) = x(2,-1,3,2) + y(-1,1,1,-3) + z(1,1,9,-5)$
comparing the terms after addition we get

$$\left.\begin{aligned} 2x - y + z &= 3 \\ -x + y + z &= -1 \\ 3x + y + 9z &= 0 \\ 2x - 3y - 5z &= 1 \end{aligned}\right\}$$

This is system of linear equations in x, y, z if unique solution exist for any '3' and 4th is satisfied by it then Yes V_4 will be spanned by $\{V_1, V_2, V_3\}$.

Matrix form of first 3 equations $AX = B$ is where

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 1 \\ 3 & 1 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\det(A) = 2(8) + 1(-10) + 3(-2) = 16 - 10 - 6 = 0$
clearly $\det A$ is '0' i.e $\rho(A)$ is not 3 so unique solution to the system of linear Equations does not exist.

Consider $[A|B] = \left[\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ -1 & 1 & 1 & -1 \\ 3 & 1 & 9 & 0 \end{array}\right]$

apply $r_1 \to 2r_2 + r_1$
$r_3 \to 3r_2 + r_3$

$[A|B] = \left[\begin{array}{ccc|c} 0 & 1 & 3 & 1 \\ -1 & 1 & 1 & -1 \\ 0 & 4 & 12 & -3 \end{array}\right]$

now apply $r_3 \to 4r_1 - r_3$

$$\left[\begin{array}{ccc|c} 0 & 1 & 3 & 1 \\ -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & -7 \end{array}\right]$$

clearly $\rho(A) = 2$
$\rho(A|B) = 3$

and $\rho(A) \neq \rho(A|B)$ So No Solution to system exist. Hence V_4 is not a linear span of $\{V_1, V_2, V_3\}$.

1. (b) Given Linear Transformation $T: R^3 \to R^3$ as
$T(x, y, z) = (x+z,\ x+y+2z,\ 2x+y+3z)$.
for any $\alpha(x,y,z) \in R^3$ is transformed by T in R^3.
let us take basis of R^3 $e_1(1,0,0)$ $e_2(0,1,0)$
$e_3(0,0,1)$.
$T(e_1) = (1, 1, 2)$ $T(e_2) = (0, 1, 1)$
$T(e_3) = (1, 2, 3)$
Now to find the basis of T let us convert $T(e_1)$ $T(e_2)$ $T(e_3)$ in row reduced echolon form.

let $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ $r_3 \to r_3 - r_1$ $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

now apply $r_3 \to r_3 - r_2$ $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Now A is in row-reduced echolon form so
Basis of T can be given as $\{(1,1,2), (0,1,1)\}$.
i.e $\mathrm{Dim}(T) = 2$.
$\therefore$ from Rank-nullity Theorem $\rho(T) + \nu(T) = \dim(R^3)$.
To find nullity, let us Take $\alpha \in R^3$ such that For any $\alpha(x,y,z)$, $T(\alpha)$
$T(\alpha) = \hat{0}$ $\hat{0} \in R^3$ $(0,0,0)$.
i.e $x+z=0 \Rightarrow \boxed{x=-z}$; $x+y+2z=0 \Rightarrow \boxed{y=-z}$
then 3rd $2x+y+3z=0$ for $x=-z$ & $y=-z$.
So $\alpha(x,y,z) = (-z, -z, z) = z(-1,-1,0)$
i.e nullity is '1'.
So $\rho(T) = \dim(T) - \nu(T) = 3-1 = 2$
which is what we got so Rank-nullity Theorem also verified.

(OR)

Given that $T: \mathbb{R}^3 \to \mathbb{R}^3$ s.t
$T(x, y, z) = (x+z,\ x+y+2z,\ 2x+y+3z)$.

clearly $\dim(\mathbb{R}^3) = 3$.

Let $N(T) = \{(x,y,z) \in \mathbb{R}^3 / T(x,y,z) = (0,0,0)\}$.

be a nullspace of T.

we have

$$\left.\begin{array}{l} x + z = 0 \\ x + y + 2z = 0 \\ 2x + y + 3z = 0. \end{array}\right\} \Rightarrow AX = 0$$

where $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$,

we have.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

clearly it is in echelon form.

we have $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow x + z = 0 \Rightarrow x = -z$

$y + z = 0 \Rightarrow y = -z$

clearly there is only one free variable say z.

$\therefore \nu(T) = \dim(N(T)) = 1$.

$\therefore N(T) = \{(-z, -z, z) / z \in \mathbb{R}\}$.

We know that

$$\rho(T) + \nu(T) = \dim \mathbb{R}^3$$

i.e $\dim(R(T)) + \dim(N(T)) = \dim \mathbb{R}^3$

$\Rightarrow \dim(R(T)) + 1 = 3$

$\Rightarrow \boxed{\rho(T) = \dim(R(T)) = 3-1 = 2}$

16 Q4:

Find the rank and nullity of the linear transformation:

$T: \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T(x, y, z) = (x+z, x+y+2z, 2x+y+3z)$$

1(c) Determine the values of p and q for which $\lim_{x\to 0} \dfrac{x(1+p\cos x) - q\sin x}{x^3}$ exists and equals 1.

Sol^n: It is a $\frac{0}{0}$ form and so by L Hospital's rule

$$\lim_{x\to 0} \frac{x(1+p\cos x) - q\sin x}{x^3}$$

$$= \lim_{x\to 0} \frac{(1+p\cos x) - px\sin x - q\cos x}{3x^2}$$

$$= \lim_{x\to 0} \frac{1+(p-q)\cos x - px\sin x}{3x^2} \left(\frac{1+p-q}{0}\right)$$

To get the required limit, we take $1+p-q=0$

Thus $\lim_{x\to 0} \dfrac{1+(p-q)\cos x - px\sin x}{3x^2} \left(\dfrac{0}{0}\right)$

$$= \lim_{x\to 0} \frac{-(p-q)\sin x - p\sin x - px\cos x}{6x} \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{-(p-q)\cos x - 2p\cos x + px\sin x}{6}$$

$$= \frac{-p+q-2p}{6} = \frac{-3p+q}{6}$$

we are given $\dfrac{-3p+q}{6} = 1 \Rightarrow -3p+q=6$

Also $1+p-q=0$ and $-6-3p+q=0$

On solving these equations for p and q, we obtain $p = -5/2$, $q = -3/2$

1(d) Examine the convergence of the integral $\int_0^1 \frac{\log x}{1+x} dx$

Sol^n: Since $\frac{\log x}{1+x}$ is negative on $(0,1]$,

we take $f(x) = -\frac{\log x}{1+x}$

Here 0 is the only point of infinite discontinuity of f on $[0,1]$.

Take $g(x) = \frac{1}{x^n}$

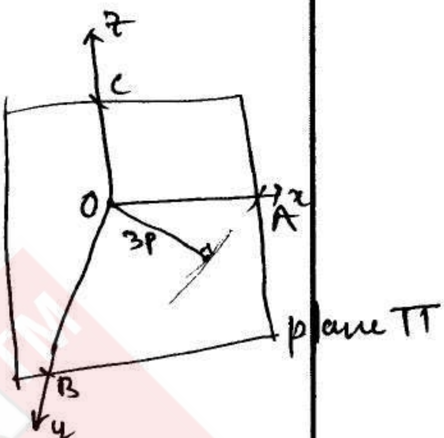
$$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} -\frac{x^n \log x}{1+x} = 0 \text{ if } n>0.$$

Taking n between 0 and 1, the integral $\int_0^1 g(x)dx$ is convergent.

$\therefore$ By comparison test, $\int_0^1 f(x)\,dx$ is convergent.

Hence $\int_0^1 \frac{\log x}{1+x} dx$ is convergent.

1.(e) Given a plane Π at a distance $3p$ from origin. The plane cuts the axes x, y, z at points A, B, C respectively.



So let the cordinates of $A(a,0,0)$ $B(0,b,0)$ $C(0,0,c)$

Then equation of any plane then plane having A, B, C as intercepts

$$\Pi: \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Now given the perpendicular distance from origin to plane is $3p$ i.e $\frac{\left|\frac{0}{a}+\frac{0}{b}+\frac{0}{c}-1\right|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}} = 3p$

or $1 = 3p\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}$ —(i)

Now let the locus of centroid of tertrahedron OABC is (α, β, γ). Also we know that centroid is $\left(\frac{a}{4}, \frac{b}{4}, \frac{c}{4}\right)$ from given points O, A, B, C.

So $\alpha = \frac{a}{4}$, $\beta = \frac{b}{4}$, $\gamma = \frac{c}{4}$ or $a = 4\alpha$, $b = 4\beta$, $c = 4\gamma$.

Now lets us put the values of (a, b, c) in (i)

$$\frac{1^2}{9p^2} = \frac{1}{(4\alpha)^2} + \frac{1}{(4\beta)^2} + \frac{1}{(4\gamma)^2} \Rightarrow \frac{16}{9p^2} = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$$

$\because \alpha, \beta, \gamma$ is coordinates of locus of centroid so replace $\alpha \to x$, $\beta \to y$, $\gamma \to z$ we get $\frac{16}{p^2} = 9\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)$

Ans.

2.(a) Given $T: R^3 \to R^3$ is a linear Transformation and $A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$ is matrix of L.T. relative to $e_1(1,0,0)$ $e_2(0,1,0)$ $e_3(0,0,1)$

i.e $\{e_1, e_2, e_3\}$ is standard basis of R^3.

So $T(e_1) = 1(e_1) + -1(e_2) + 0(e_3)$
$= (1,-1,0)$

$T(e_2) = 1(e_1) + 2(e_2) + 1(e_3)$
$= (1,2,1)$

$T(e_3) = 2(e_1) + 1(e_2) + 3(e_3)$
$= (2,1,3)$.

So let any $\alpha(x,y,z) \in R^3$ can be expressed as $\alpha = xe_1 + ye_2 + ze_3$

So $T(\alpha) = T(xe_1 + ye_2 + ze_3) = xT(e_1) + y(T(e_2)) + zT(e_3)$.
$= x(1,-1,0) + y(1,2,1) + z(2,1,3)$

$\boxed{T(x,y,z) = (x+y+2z, -x+2y+z, y+3z)}$ is required linear Transformation.

Now need to find Matrix of T relative to basis $S\{(1,1,1), (0,1,1), (0,0,1)\}$ say $\{\alpha_1, \alpha_2, \alpha_3\}$.

So $T(\alpha_1) = (4, 2, 4)$ $T(\alpha_3) = (2,1,3)$
$T(\alpha_2) = (3, 3, 4)$

Now let any $\beta \in R^3$ $\beta(a,b,c)$ then where $x,y,z \in R$
$\beta = x(\alpha_1) + y(\alpha_2) + z(\alpha_3)$
$(a,b,c) = (x, x+y, x+y+z)$

or $\boxed{x = a}$ $\boxed{y = b-a}$ $\boxed{z = c-b}$

So $\beta(a,b,c) = a\alpha_1 + (b-a)\alpha_2 + (c-b)\alpha_3$ as linear combination.

2(a) continue

Now we have

$T(\alpha_1) = (4,2,4) = 4\alpha_1 + (-2)\alpha_2 + 2\alpha_3$

$T(\alpha_2) = (3,3,4) = 3\alpha_1 + 0\cdot\alpha_2 + 1\alpha_3$

$T(\alpha_3) = (2,1,3) = 2\alpha_1 + (-1)\alpha_2 + 2\alpha_3$

So matrix of linear Transformation for Basis $\{\alpha_1\ \alpha_2\ \alpha_3\}$ is

$$[T:S] = \begin{bmatrix} 4 & 3 & 2 \\ -2 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix}$$

2(c) Qn: If the matrix of a linear transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ relative to the basis $\{(1,0,0),(0,1,0),(0,0,1)\}$ is $\begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$.

then find of relative to the basis $\{(1,1,1),(0,1,1),(0,0,1)\}$.

2(b) Evaluate the triple integral which gives the volume of the solid between the two paraboloids $z = 5(x^2+y^2)$ and $z = 6-7x^2-y^2$.

Sol^n: The bounds for z are given by $z = 5x^2+5y^2$ to $z = 6-7x^2-y^2$.

projecting this region bounded by the two paraboids curve of intersection namely, $5x^2+5y^2 = 6-7x^2-y^2$

$$\Rightarrow 2x^2+y^2=1$$

Thus the volume of the region

$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} \int_{5x^2+5y^2}^{6-7x^2-y^2} 1\,dz\,dy\,dx$$

By using the cylindrical co-ordinates

$x = \frac{r}{\sqrt{2}}\cos\theta$, $y = r\sin\theta$ and $z = z$

This reduces the elliptic region to $r=1$.

It is easy to check that the Jacobian of this transformation is equal to $\frac{r}{\sqrt{2}}$

Hence, the change of variables

$$V = \int_0^{2\pi}\int_0^1\int_{\frac{5}{2}r^2\cos^2\theta+5r^2\sin^2\theta}^{6-\frac{7}{2}r^2\cos^2\theta-r^2\sin^2\theta} 1.\ \frac{r}{\sqrt{2}}\,dz\,dr\,d\theta$$

$$= \frac{1}{\sqrt{2}}\int_0^{2\pi}\int_0^1 (6-6r^2)\,r\,dr\,d\theta$$

$$= \frac{1}{\sqrt{2}}\,2\pi\int_0^1 (6-6r^2)\,r\,dr$$

$$= \frac{2\pi}{\sqrt{2}}\left(\frac{6r^2}{2}-\frac{6r^4}{4}\right)_0^1$$

$$= \frac{2\pi}{\sqrt{2}}\left(3-\frac{3}{2}\right)$$

$$= \frac{2\pi}{\sqrt{2}}\,\frac{3}{2}$$

$$= \frac{3\pi}{\sqrt{2}}.$$

2c(i) →

Given equation: $2x^2+3y^2-8x+6y-12z+11=0$

Here $a=2, b=3, c=0$

$f=g=h=0$

$u=-4, v=3, w=-6$

$d=11$

∴ discriminating cubic is:

$$① \equiv \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda(\lambda-2)(\lambda-3)=0$$

$$\Rightarrow \lambda_1 = 2,\ \lambda_2=3,\ \lambda_3=0 \text{ (let say)}$$

Now putting $\lambda=0$ in the determinant given by ① & associating each row with l_3, m_3, n_3, we have

$2l_3=0 \Rightarrow l_3=0$

$3m_3=0 \Rightarrow m_3=0$

∴ (l_3, m_3, n_3) are D.C's of axis corresponding to $\lambda_3=0$,

∴ $\cancel{l_3^2}+\cancel{m_3^2}+n_3^2=1 \Rightarrow n_3=1$

Also, $k = ul_3+vm_3+wn_3$

$= -4\times 0+3\times 0-6\times 1 = -6$

∴ required reduced equation is: $\lambda_1 x^2+\lambda_2 y^2+2kz=0$

$\Rightarrow 2x^2+3y^2-12z=0$

Which clearly represents an elliptic paraboloid.

For finding vertex,

$$\frac{\frac{\partial F}{\partial x}}{l_3} = \frac{\frac{\partial F}{\partial y}}{m_3} = \frac{\frac{\partial F}{\partial z}}{n_3} = k$$

$\Rightarrow \frac{\partial F}{\partial x}=0 \Rightarrow 4x-8=0 \Rightarrow x=2$ — Ⓐ

$\Rightarrow \frac{\partial F}{\partial y}=0 \Rightarrow 6y+6=0 \Rightarrow y=-1$ — Ⓑ

$\Rightarrow \frac{\partial F}{\partial z}=-12$

& $k(l_3x+m_3y+n_3z)$
$+ux+vy+wz+d=0$

$\Rightarrow -6z-4x+3y-6z+11=0$

$\Rightarrow 4x-3y+12z=11$ — Ⓒ

From Ⓐ, Ⓑ & Ⓒ

vertex: $(x,y,z) = (2,-1,0)$

∴ $\boxed{\text{eqn. of principal axis: } \frac{x-2}{0} = \frac{y+1}{0} = \frac{z}{1}}$

Also, equation of principal planes:

$\frac{\partial F}{\partial x}=0 \Rightarrow \boxed{x-2=0}$

&

$\frac{\partial F}{\partial y}=0 \Rightarrow \boxed{y+1=0}$: principal planes

2. C(ii) → The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the coordinate axes in A,B,C respectively. Prove that the equation of the cone generated by the lines drawn from the origin O to meet the circle ABC is

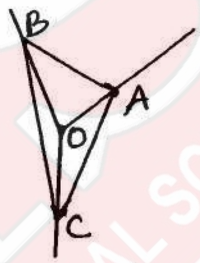
$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{b}{a}+\frac{a}{b}\right)=0.$$

Sol^n: The equation of the plane is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \longrightarrow (1)$$

since it meets the coordinate axes in A, B, C. the coordinates of A,B,C are $(a,0,0)$, $(0,b,0)$, $(0,0,c)$

Now the circle through A,B,C is the intersection of plane through A,B,C i.e. plane (1) and any sphere through the points. A,B,C say the sphere OABC.



Now the sphere OABC through the points $O(0,0,0)$, $A(a,0,0)$, $B(0,b,0)$, $C(0,0,c)$ is $x^2+y^2+z^2-ax-by-cz=0$. ——— (2)

∴ The guiding curve is the circle given by (1) & (2)

i.e., $x^2+y^2+z^2-ax-by-cz=0$;
$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$

$$(2) \equiv x^2+y^2+z^2-(ax+by+cz)(1)=0$$

$$\Rightarrow x^2+y^2+z^2-(ax+by+cz)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)(1)=0$$

$$\Rightarrow x^2+y^2+z^2-x^2-\frac{b}{a}xy-\frac{c}{a}zx-\frac{a}{b}xy-y^2-\frac{c}{b}yz$$
$$-\frac{a}{c}zx-\frac{b}{c}yz-z^2=0$$

$$\Rightarrow -yz\left(\frac{b}{c}+\frac{c}{b}\right)-zx\left(\frac{c}{a}+\frac{a}{c}\right)-xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$$

$$\Rightarrow yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$$

$$\Rightarrow \sum a\,(b^2+c^2)\,yz=0 \longrightarrow (3)$$

which is required equation of the cone.

3(c) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(i) verify the cayley-Hamilton theorem for the Matrix A.

(ii) Show that $A^n = A^{n-2} + A^2 - I$ for $n \geq 3$, where I is the identity Matrix of order '3'. Hence, find A^{40}.

Sol:

We have

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 0-\lambda & 1 \\ 0 & 1 & 0-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2 - 1) = 0.$$

$$\Rightarrow \lambda^2 - 1 - \lambda^3 + \lambda = 0$$

$$\Rightarrow -\lambda^3 + \lambda^2 + \lambda - 1 = 0$$

clearly $\Rightarrow \boxed{\lambda^3 - \lambda^2 - \lambda + 1 = 0}$

Which is the required characteristic equation of A.

since $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$,

$$A^2 = AA = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

we have

$$A^3 = A^2 \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

we have

$$A^3 - A^2 - A + I = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

clearly A satisfies its own characteristic equation.

$\therefore$ cayley-Hamilton theorem is verified.

(ii) If $n=3$, then $A^3 = A + A^2 - I$ ——— (1)

since $A^2 = A \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

NOW $A + A^2 - I = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and $A^3 = A^2 \cdot A$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$\therefore A^n = A^{n-2} + A^2 - I$ is true for $n=3$

suppose $A^n = A^{n-2} + A^2 - I$ is true for $n=k$.

$\therefore A^k = A^{k-2} + A^2 - I$

Now for $n = k+1$

$$A^{k+1} = A \cdot A^k$$
$$= A\left[A^{k-2} + A^2 - I\right]$$
$$= A^{k-1} + A^3 - A$$
$$= A^{k-1} + A + A^2 - I - A$$
$$= A^{k-1} + A^2 - I$$ is true for $n = k+1$

$\therefore A^n = A^{n-2} + A^2 - I$.

$\therefore$ By mathematical induction,
It is true for every integer $n \geq 3$.

$$A^n = A^{n-2} + A^2 - I \longrightarrow (2)$$

Now $A^3 = A + A^2 - I$

$A^4 = 2A^2 - I$

$A^6 = A^4 + A^2 - I$

$A^6 = 3A^2 - 2I$

$A^8 = 4A^2 - 3I$

$A^{10} = A^8 + A^2 - I$

$A^{10} = 5A^2 - 4I$ $\left(\text{i.e. } A^{10} = \frac{10}{2}A^2 - \left(\frac{10}{2}-1\right)I\right)$

$A^{12} = 6A^2 - 5I$

$\left(\text{i.e. } A^{12} = \frac{12}{2}A^2 - \left(\frac{12}{2}-1\right)I\right)$

$\vdots$

$$A^{40} = \frac{40}{2}A^2 - \left(\frac{40}{2}-1\right)I$$

$$= 20A^2 - 19I$$

$$= 20\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - 19\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 20 & 1 & 0 \\ 20 & 0 & 1 \end{bmatrix}$$

3(b) Justify whether (0,0) is an extreme point for the function $f(x,y) = 2x^4 - 3x^2 y + y^2$.

Sol. Given

$$f(x,y) = 2x^4 - 3x^2 y + y^2$$

$$f_x = 8x^3 - 6xy, \quad f_{xx} = 24x^2 - 6y$$

$$f_y = -3x^2 + 2y, \quad f_{yy} = 2$$

$$f_{xy} = -6x$$

At (0,0),

$$AC - B^2 = f_{xx} f_{yy} - (f_{xy})^2 = 0$$

Thus it is a doubtful case and so needs further investigation.

We have $f(0,0) = 0$

Clearly $f(x,y) = (x^2 - y)(2x^2 - y)$

$$\therefore f(x,y) - f(0,0) = (x^2 - y)(2x^2 - y)$$

$$> 0, \begin{cases} \text{for } x^2 > y, \left(\text{and so } 2x^2 > y\right) \\ \text{for } x^2 < \frac{y}{2} \left(\text{and so } x^2 < y\right) \end{cases}$$

$$< 0 \text{ for } \frac{y}{2} < x^2 < y.$$

Thus we observe that $f(x,y)-f(0,0)$ doesnot keep the same sign near the origin and so the given function has neither a maximum nor a minimum value at the origin.

3(c) Find the equation of the sphere through the circle

$$x^2+y^2+z^2-4x-6y+2z-16=0;$$
$$3x+y+3z-4=0$$ in the following two cases:

(i) the point $(1,0,-3)$ lies on the sphere

(ii) the given circle is a great circle of the sphere.

Sol: Given circle is:

$$x^2+y^2+z^2-4x-6y+2z-16=0,$$
$$3x+y+3z-4=0$$

Let the equation of sphere through given circle be:

$$x^2+y^2+z^2-4x-6y+2z-16+\lambda(3x+y+3z-4)=0 \quad (1)$$

(i) if the point $(1,0,-3)$ lies on the sphere (1) then

$$1+0+9-4+0-6-16+\lambda(3+0-9-4)=0.$$

$$\Rightarrow -16+\lambda(-10)=0$$

$$\Rightarrow \lambda=-\frac{16}{10} \Rightarrow \boxed{\lambda=-8/5}$$

$\therefore (x^2+y^2+z^2-4x-6y+2z-16)$
$-\frac{8}{5}(3x+y+3z-4)=0$

$\Rightarrow x^2+y^2+z^2-\left(4+\frac{24}{5}\right)x-\left(6+8/5\right)y$
$+\left(2-\frac{24}{5}\right)z-16+\frac{32}{5}=0.$

$\Rightarrow \boxed{x^2+y^2+z^2-\frac{44}{5}x-\frac{38}{5}y-\frac{14}{5}z-\frac{48}{5}=0.}$

which is the required equation of the sphere.

(ii) from (1),

$x^2+y^2+z^2+(-4+3\lambda)x+(-6+\lambda)y$
$+(2+3\lambda)z+(16-4\lambda)=0.$ — (I)

It's centre

$\left(\frac{+4-3\lambda}{2}, \frac{6-\lambda}{2}, \frac{-2-3\lambda}{2}\right).$

Since it lies on the plane
$3x+y+3z-4=0$

$\Rightarrow 3\left(\frac{4-3\lambda}{2}\right)+\left(\frac{6-\lambda}{2}\right)+3\left(-\frac{2-3\lambda}{2}\right)-4=0$

$\Rightarrow 6-\frac{9}{2}\lambda+\cancel{3}-\frac{\lambda}{2}-\cancel{3}-\frac{9\lambda}{2}-4=0.$

$\Rightarrow 2-\frac{19}{2}\lambda=0$

$\Rightarrow \frac{19}{2}\lambda=2 \Rightarrow \boxed{\lambda=\frac{4}{19}}$

3(C) continuity:

$\therefore$ (I) $\equiv$

$$x^2+y^2+z^2+\left(-4+\frac{12}{19}\right)x+\left(-6+\frac{4}{19}\right)y+\left(2+\frac{12}{19}\right)z+\left(-16-\frac{16}{19}\right)=0.$$

$$\Rightarrow x^2+y^2+z^2+\left(-\frac{64}{19}\right)x+\left(-\frac{110}{19}\right)y+\frac{50}{19}z-\frac{320}{19}=0.$$

which is the required sphere.

4(k) Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ -1 & 3 & 0 & -4 \\ 2 & 1 & 3 & -2 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$ by using it to row-reduced echelon form.

Sol: $A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ -1 & 3 & 0 & -4 \\ 2 & 1 & 3 & -2 \\ 1 & 1 & 1 & -1 \end{bmatrix}$

$R_2 \to R_2 + R_1$
$R_3 \to R_3 - 2R_1$
$R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 5 & -1 & -4 \\ 0 & -3 & 5 & -2 \\ 0 & -1 & 2 & -1 \end{bmatrix}$$

$R_2 \to \frac{1}{5} R_2$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -1/5 & -4/5 \\ 0 & -3 & 5 & -2 \\ 0 & -1 & 2 & -1 \end{bmatrix}$$

$R_3 \to R_3 + 3R_2$
$R_4 \to R_4 + R_2$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -1/5 & -4/5 \\ 0 & 0 & \frac{22}{5} & -\frac{22}{5} \\ 0 & 0 & \frac{9}{5} & -\frac{9}{5} \end{bmatrix}$$

$R_3 \to \frac{5}{22} R_3$
$R_4 \to \frac{5}{9} R_4$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -1/5 & -4/5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -1/5 & -4/5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_4 \to R_4 - R_3$$

$$\sim \begin{bmatrix} 1 & 0 & -3/5 & 8/5 \\ 0 & 1 & -1/5 & -4/5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_1 \to R_1 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1/5 & -4/5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_1 \to R_1 + \tfrac{3}{5}R_3$$

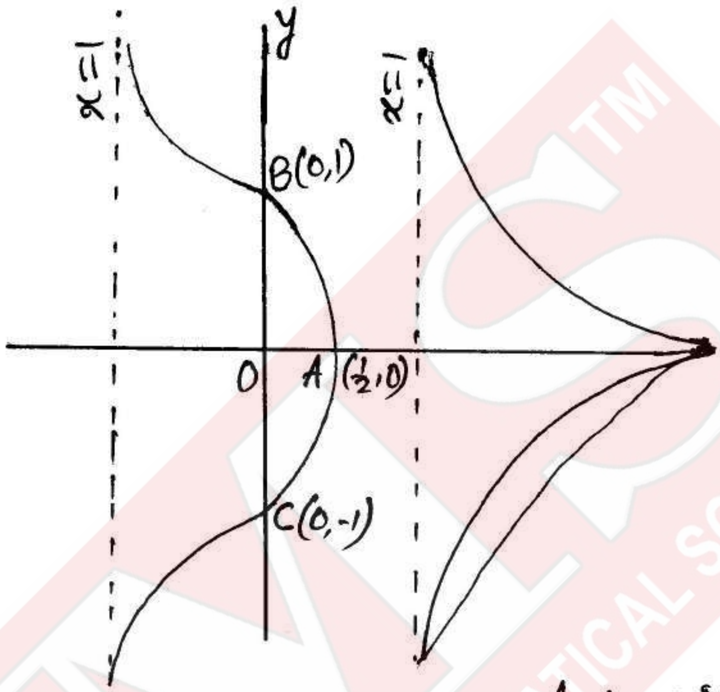
$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 \to R_2 + \tfrac{1}{5}R_3$$

clearly it is in row-reduced echelon form.
and has three non-zero rows

$\therefore \boxed{\rho(A) = 3}$

4(b) → Trace the curve $y^2(x^2-1)=2x-1$.

Solⁿ: The curve is symmetrical about the x-axis. It does not pass through the origin.



The curve meets the x-axis at the point $A(\frac{1}{2},0)$ and the y-axis at the point $B(0,1)$ and $C(0,-1)$.

The asymptotes parallel to the y-axis are $x^2-1=0$ or $x=\pm 1$ and that parallel to the x-axis is $y=0$ (i.e., x-axis). It may be seen that $y^2<0$ in the region $\frac{1}{2}<x<1$ [take, for ex. $x=\frac{2}{3}$ in $y^2=(2x-1)/(x^2-1)$].

Thus the curve does not lie in the region $\frac{1}{2}<x<1$. Hence the graph of the curve is as shown in the above figure.

4.(c) Prove that the locus of a line which meets the lines $y=\pm mx$, $z=\pm c$ and the circle $x^2+y^2=a^2$, $z=0$ is $c^2m^2(cy-mzx)^2+c^2(yz-cmx)^2=a^2m^2(z^2-c^2)^2$.

Sol'n: The given lines are

$$y-mx=0,\quad z-c=0 \quad \text{——— (1)}$$

$$y+mx=0,\quad z+c=0 \quad \text{——— (2)}$$

and the circle is

$$x^2+y^2=a^2;\quad z=0 \quad \text{——— (3)}$$

Any line intersecting (1) & (2) is

$$\left.\begin{array}{l} y-mx+k_1(z-c)=0 \\ y+mx+k_2(z+c)=0 \end{array}\right\} \quad \text{——— (4)}$$

If it meets the circle (3), we have to eliminate x, y, z from (3) & (4).

(3) ≡ Putting $z=0$ in (4), we get

$$y-mx+k_1(c)=0$$

$$y+mx+k_2(c)=0$$

Solving

$$\frac{y}{-mk_2c+mk_1c}=\frac{x}{-ck_1-ck_2}=\frac{1}{m+m}$$

$$\Rightarrow x=\frac{-(k_1+k_2)c}{2m}$$

$$y=\frac{c(k_1-k_2)}{2}$$

Putting these values of x, y in (3), we get

$$\frac{c^2(k_1+k_2)^2}{4m^2}+\frac{c^2(k_1-k_2)^2}{4}=a^2$$

$$\Rightarrow c^2(k_1+k_2)^2+c^2m^2(k_1-k_2)^2=4a^2m^2 \quad \text{——— (5)}$$

To find the locus,

eliminate k_1, k_2 from ④ & ⑥

$$\therefore ④ \equiv k_1 = \frac{-(y-mx)}{z-c} = \frac{mx-y}{z-c}$$

$$k_2 = \frac{-(y+mx)}{z+c}$$

Substituting these values in ⑤

$$\therefore ⑤ \equiv c^2\left[\left(\frac{mx-y}{z-c}\right)+\left(\frac{-mx-y}{z+c}\right)\right]+c^2m^2\left[\left(\frac{mx-y}{z-c}\right)+\left(\frac{mx+y}{z+c}\right)\right]$$

$$= 4a^2m^2$$

on simplification we get

$$c^2m^2(cy-mxz)^2+c^2(yz-mx)^2 = a^2m^2(z^2-c^2)^2$$

which is the required locus.

5.(a) Obtain the solution of the initial-value problem $\frac{dy}{dx} - 2xy = 2$, $y(0)=1$ in form of
$y = e^{x^2}\left[1+\sqrt{\pi}\,\text{erf}(x)\right]$.

Sol^n: The given ODE is linear which is

$\frac{dy}{dx} - 2xy = 2$ [LODE in y].

To get the general solution of it we need to find the integrating factor (I.F.) such that general solution is

$y \cdot I.F = \int 2\, I.F\, dx + C$

so $I.F = e^{\int P(x)\,dx}$ here $P(x) = -2x$.

so $I.F = e^{\int -2x\,dx} = e^{-x^2}$

Now g.s. is

$y \cdot e^{-x^2} = \int 2e^{-x^2}dx + C$

where c is ~~para~~ integrating constant.

$\because y(0)=1 \Rightarrow 1\cdot e^0 = 2\int e^{-x^2}dx + C$

$\because dx = 0$ for $x=0$ $\Rightarrow 1 = 0 + C \Rightarrow \boxed{C=1}$

Now $y = e^{x^2}\left[1 + 2\int e^{-x^2}dx\right]$

$\because \int e^{-x^2}dx = \int_0^x e^{-u^2}du$.

And we know that erf(x) is defined as

$\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du$ so $2\int_0^x e^{-u^2}du = \sqrt{\pi}\,\text{erf}(x)$.

Now replace $2\int_0^x e^{-u^2}du$ we get

$y = e^{x^2}\left[1+\sqrt{\pi}\,\text{erf}(x)\right]$ Ans.

5(b) Show that $\int_0^\infty \frac{f(t)}{t}dt = \int_0^\infty F(x)\,dx$. Hence evaluate the integral $\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}dt$.

Sol^n: Let $G(t) = \frac{1}{t}f(t)$

i.e $f(t) = tG(t)$

$$\therefore L\{f(t)\} = L\{tG(t)\} = -\frac{d}{dp}L\{G(t)\}$$

[By the theorem

If $F(t)$ is a function of class A and if $L\{F(t)\} = f(p)$

then $L\{tF(t)\} = -f'(p)$]

$$\Rightarrow f(p) = -\frac{d}{dp}L\{G(t)\}$$

Now integrating both the sides w.r.t p from p to ∞, we have

$$-\left[L\{G(t)\}\right]_p^\infty = \int_p^\infty f(p)\,dp$$

$$\Rightarrow -\lim_{p\to\infty} L\{G(t)\} + L\{G(t)\} = \int_p^\infty f(p)\,dp$$

$$\Rightarrow 0 + L\{G(t)\} = \int_p^\infty f(p)\,dp$$

$$\left[\because \lim_{p\to\infty} L\{G\{t\}\} = \lim_{p\to\infty}\int_0^\infty e^{-pt}G(t)\,dt = 0\right]$$

$$\Rightarrow L\left\{\frac{1}{t}f(t)\right\} = \int_p^\infty F(x)\,dx \quad \text{proved.}$$

Let $f(t) = e^{-t} - e^{-3t}$

$\therefore f(p) = L\{f(t)\} = L\{e^{-t}\} - L\{e^{-3t}\}$

$$= \frac{1}{p+1} - \frac{1}{p+3}$$

$$\therefore L\left\{\frac{f(t)}{t}\right\} = \int_p^\infty F(x)\,dx$$

$$\Rightarrow \int_0^\infty e^{-pt}\frac{(e^{-t} - e^{-3t})}{t}\,dt = \int_p^\infty \left(\frac{1}{x+1} - \frac{1}{x+3}\right)dx$$

$$= -\log\frac{p+1}{p+3}$$

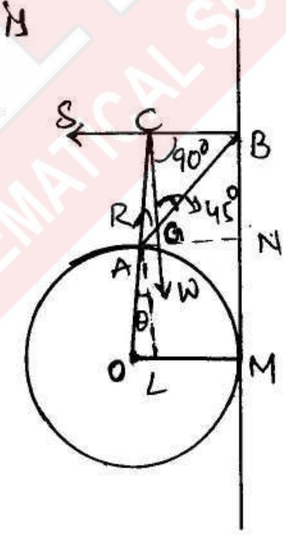
$$= \log\frac{p+3}{p+1}$$

$\therefore$ Taking limit as $p \to 0$, we have

$$\int_0^\infty \frac{e^{-t} - e^{-3t}}{t}\,dt = \log\frac{3}{1} = \log 3$$

5(c) A cylinder of radius 'a' touches a vertical wall along a generating line. Axis of the cylinder is fixed horizontally. A uniform flat beam of length 'l' and weight 'W' rests with its extremities in contact with the wall and the cylinder, making an angle of 45° with the vertical. If frictional forces are neglected, then show that $\frac{a}{l} = \frac{\sqrt{5}+5}{4\sqrt{2}}$.
Also, find the reactions of the cylinder and wall.

The figure represents one vertical section through the beam AB. The beam is in equilibrium under the action of three forces.



1. its weight W acting vertically downwards through G, its mid-point
2. The normal reaction R at A
3. the normal reaction S at B.

They must meet in a point (say C)

$\angle CGB = 45^\circ$ (given) $\angle BCG = 90$

Let $\angle ACG = \theta$, $AG = GB = \frac{l}{2}$

By m-n theorem in ΔABC

$$\left(\frac{l}{2}+\frac{l}{2}\right)\cot 45^\circ = \frac{l}{2}\cot\theta - l\cot 90$$

$$\Rightarrow \cot\theta = 2$$

$$\therefore \cos\theta = \frac{2}{\sqrt5}, \quad \sin\theta = \frac{1}{\sqrt5}$$

By Lami's theorem to forces at C, we have

$$\frac{R}{\sin 90} = \frac{S}{\sin(180-\theta)} = \frac{W}{\sin(90+\theta)}$$

$$\frac{R}{1} = \frac{S}{\sin\theta} = \frac{W}{\cos\theta}$$

$$R = \frac{W}{\cos\theta} = \frac{\sqrt5 W}{2}$$ which is the reaction of the cylinder

& $S = W\frac{\sin\theta}{\cos\theta} = \frac{W}{2}$ which is the reaction of the wall

Now $OM = OL + LM = OL + AN$

$$a = OA\sin\theta + AB\sin 45$$

$$= a\sin\theta + l\frac{1}{\sqrt2}.$$

$$a = \frac{a}{\sqrt5} + \frac{l}{\sqrt2}$$

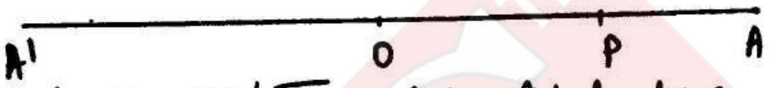
$$\left(\frac{\sqrt5-1}{\sqrt5}\right)a = \frac{l}{\sqrt2}$$

$$\frac{a}{l} = \frac{\sqrt5}{\sqrt2}\frac{1}{(\sqrt5-1)}$$

$$\frac{a}{l} = \frac{\sqrt5}{\sqrt2}\frac{\sqrt5+1}{4} = \frac{5+\sqrt5}{4\sqrt2}$$

5(d) A particle is performing a simple harmonic motion of period T about a centre O and it passes through a point P with velocity v along the direction OP and OP = P. Find the time that elapses before the particle returns to the point P. What will be the value of p when the elapsed time is $T/2$?

Sol'n Let the eq^n of the S.H.M with centre O as origin be $\frac{d^2x}{dt^2} = -\mu x$



The time period $T = 2\pi/\sqrt{\mu}$, let aplitude be a

Then $(dx/dt)^2 = \mu(a^2 - x^2)$ ——(1)

When particle passes through P its velocity is given to be V in the direction OP. Also $OP = b$. So putting $x = b$ and $dx/dt = v$ in (1)

We get $V^2 = \mu(a^2 - b^2)$.

In S.H.M the time from P to A is equal to the time from A to P.

$\therefore$ the required time $= 2 \cdot$ time from A to P.

Now motion from A to P, we have

$\frac{dx}{dt} = -\sqrt{\mu}\sqrt{(a^2 - x^2)} \Rightarrow dt = -\frac{1}{\sqrt{\mu}} \frac{dx}{\sqrt{(a^2 - x^2)}}$

Let t_1 be the time from A to P. Then at A, $t = 0$, $x = a$ and at P, $t = t_1$ and $x = b$, Therefore integrating (3)

we get $\int_0^{t_1} dt = \frac{1}{\sqrt{\mu}} \int_a^b \frac{-dx}{\sqrt{(a^2 - x^2)}} \Rightarrow t_1 = \frac{1}{\sqrt{\mu}} \left[\cos^{-1}\frac{x}{a}\right]_a^b$

Hence required time $= 2t_1 = \frac{2}{\sqrt{\mu}} \cos^{-1}\left(\frac{b}{a}\right)$

$= \frac{2}{\sqrt{\mu}} \tan^{-1}\left\{\frac{\sqrt{(a^2 - b^2)}}{b}\right\} = \frac{2}{\sqrt{\mu}} \tan^{-1}\left(\frac{V}{b\sqrt{\mu}}\right)$ [from (2)

$= \frac{2}{2\pi/T} \tan^{-1}\left\{\frac{V}{b(2\pi/T)}\right\}$ [$\because T = 2\pi/\sqrt{\mu}$ so that $\sqrt{\mu} = 2\pi/T$]

$= \frac{T}{\pi} \tan^{-1}\left(\frac{VT}{2\pi b}\right)$

5(e) If $\vec{a} = \sin\theta\, i + \cos\theta\, j + \theta k$

$\vec{b} = \cos\theta\, i - \sin\theta\, j - 3k$

$\vec{c} = 2i + 3j - 3k$

~~then~~ find the values of the derivative of the vector function $\vec{a} \times (\vec{b} \times \vec{c})$ w.r.t θ at $\theta = \pi/2$ and $\theta = \pi$.

sol:

We have
$$\vec{b} \times \vec{c} = \begin{vmatrix} i & j & k \\ \cos\theta & -\sin\theta & -3 \\ 2 & 3 & -3 \end{vmatrix}$$

$$= i(3\sin\theta + 9) - j(-3\cos\theta + 6) + k(3\cos\theta + 2\sin\theta)$$

We have
$$\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} i & j & k \\ \sin\theta & \cos\theta & \theta \\ 3\sin\theta + 9 & 3\cos\theta - 6 & 3\cos\theta + 2\sin\theta \end{vmatrix}$$

$$= i\left[3\cos^2\theta + 2\sin\theta\cos\theta - 3\theta\cos\theta + 6\theta\right]$$
$$- j\left[3\sin\theta\cos\theta + 2\sin^2\theta - 3\theta\sin\theta - 9\theta\right]$$
$$+ k\left[3\sin\theta\cos\theta - 6\sin\theta - 3\cos\theta\sin\theta - 9\cos\theta\right].$$

let $\vec{f} = \vec{a} \times (\vec{b} \times \vec{c})$

$$= i\left[3\cos^2\theta + \sin 2\theta - 3\theta\sin\theta + 6\theta\right]$$
$$-j\left[\tfrac{3}{2}\sin 2\theta + 2\sin^2\theta - 3\theta\sin\theta - 9\theta\right]$$
$$+k\left[\tfrac{3}{2}\sin 2\theta - 6\sin\theta - \tfrac{3}{2}\sin 2\theta - 9\cos\theta\right].$$

we have

$$\frac{d\vec{f}}{d\theta} = i\left[-6\cos\theta\sin\theta + 2\cos 2\theta - 3\theta\cos\theta - 3\sin\theta + 6\right]$$
$$-j\left[3\cos 2\theta + 4\sin\theta\cos\theta - 3\sin\theta - 3\theta\cos\theta - 9\right]$$
$$+k\left[3\cos 2\theta - 6\cos\theta - 3\cos 2\theta + 9\sin\theta\right]$$

at $\theta = \pi/2$:

$$\frac{d\vec{f}}{d\theta} = i\left[-2-3+6\right] - j\left[-3-3-9\right] + k\left[-3-3+9\right].$$
$$= i + 15j + 3k.$$

at $\theta = \pi$:

$$\frac{d\vec{f}}{d\theta} = i\left[2+3\pi+6\right] - j\left[3+3\pi-9\right] + k\left[-3+6-3\right]$$
$$= i\left[8+3\pi\right] - j\left[3\pi-6\right] + 0k.$$

6(a) Solve the differential equation: $\frac{d^3y}{dx^3}-3\frac{d^2y}{dx^2}+4\frac{dy}{dx}-2y=e^x+\cos x$.

Sol^n: Given $(D^3-3D^2+4D-2)y=e^x+\cos x$ —— (1)

where $D\equiv d/dx$

Its auxiliary equation is $D^3-3D^2+4D-2=0$

$\Rightarrow D^2(D-1)-2D(D-1)+2(D-1)=0$

$\Rightarrow (D-1)(D^2-2D+2)=0$

$\Rightarrow D=1,\ 1\pm i$

$\therefore$ C.F $=c_1e^x+e^x(c_2\cos x+c_3\sin x)$, c_1,c_2,c_3 being arbitrary constants.

P.I corresponding to $e^x=\frac{1}{D^3-3D^2+4D-2}e^x$

$$=\frac{1}{(D-1)(D^2-2D+2)}e^x$$

$$=\frac{1}{D-1}\frac{1}{1-2+2}e^x$$

$$=\frac{1}{D-1}e^x\cdot 1=e^x\frac{1}{(D+1)-1}\cdot 1$$

$$=e^x\frac{1}{D}\cdot 1=xe^x$$

P.I corresponding to $\cos x=\frac{1}{D^3-3D^2+4D-2}\cos x$

$$=\frac{1}{D^2\cdot D-3D^2+4D-1}\cos x$$

$$=\frac{1}{(-1)^2D-3(-1)^2+4D-2}\cos x$$

$$= \frac{1}{3D+1} \cos x$$

$$= (3D-1)\frac{1}{9D^2-1} \cos x$$

$$= (3D-1)\frac{1}{9(-1^2)-1} \cos x$$

$$= -\frac{1}{10}(3D\cos x - \cos x)$$

$$= -\frac{1}{10}(-3\sin x - \cos x)$$

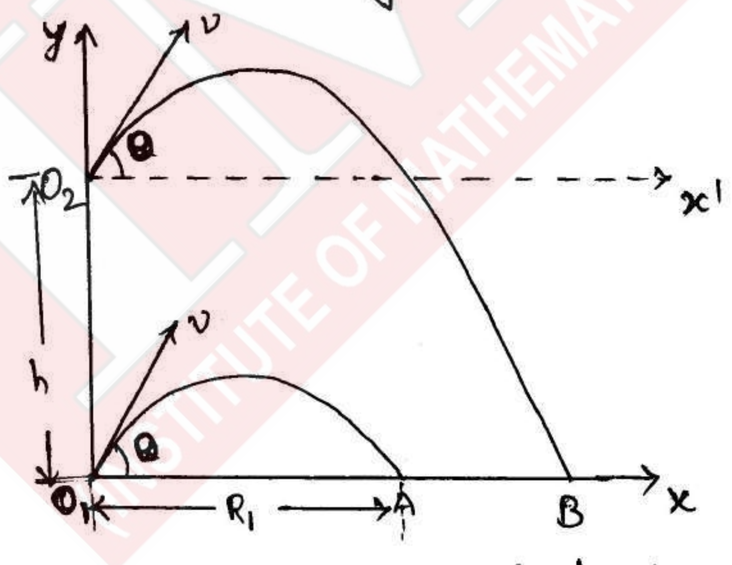
$\therefore$ Required solution is

$$y = e^x(C_1 + C_2\cos x + C_3\sin x) + xe^x + (3\sin x + \cos x)/10.$$

6(b) When a particle is projected from a point O_1 on the sea level with a velocity v and angle of projection θ with the horizon in a vertical plane, its horizontal range is R_1. If it is further projected from a point O_2, which is vertically above O_1 at a height h in the same vertical plane, with the same velocity v and same angle θ with the horizon, its horizontal range is R_2. Prove that $R_2 > R_1$ and $(R_2 - R_1) : R_1$ is equal to $\frac{1}{2}\left\{\sqrt{\left(1+\frac{2gh}{v^2\sin^2\theta}\right)}-1\right\} : 1$.

Sol^n: Let R_1 be the original horizontal range.

Then $O_1A = R_1 = \dfrac{2v^2 \sin\theta\cos\theta}{g}$ ——— ①.



Let O_2 be a point at a height h above the sea level. Let $OB = R_2$ be the new horizontal range on the sea. When the shot is fired from O_2. The equation of the new trajectory (on shifting the origin to $(0,h)$) referred to OXY as axes is

$$y = h + \theta \tan\theta - \frac{1}{2}\frac{gx^2}{v^2\cos^2\theta}$$

$[\because x = x' + 0 \Rightarrow x' = x$ and $y = y' + h \Rightarrow y' = y - h]$

Since the point $B(R_2, 0)$ lies on the above trajectory, therefore

$$0 = h + R_2 \tan\theta - \frac{1}{2}\frac{gR_2^2}{v^2\cos^2\theta}$$

$$\Rightarrow R_2^2 - \left[\frac{2v^2}{g}\sin\theta\cos\theta\right]R_2 - \frac{2v^2h}{g}\cos^2\theta = 0 \quad \text{—— (2)}$$

Using (1) in (2), we get

$$R_2^2 - R_1R_2 - \frac{2v^2h}{g}\cos^2\theta = 0$$

$$\Rightarrow \left(R_2 - \frac{1}{2}R_1\right)^2 = \frac{1}{4}R_1^2 + \frac{2v^2h}{g}\cos^2\theta$$

$$= \frac{R_1^2}{4}\left[1 + \frac{8v^2h}{R_1^2 g}\cos^2\theta\right] = \frac{R_1^2}{4}$$

$$= \frac{R_1^2}{4}\left[1 + \frac{8v^2h\cos^2\theta}{4v^2\sin^2\theta\cos^2\theta}\cdot g\right], \text{using (1)}$$

$$= \frac{R^2}{4}\left[1 + \frac{2gh}{v^2\sin^2\theta}\right]$$

$$\therefore R_2 - \frac{1}{2}R_1 = \frac{1}{2}R_1\left[1 + \frac{2gh}{v^2\sin^2\theta}\right]^{1/2}.$$

$$\Rightarrow R_2 - R_1 = \frac{1}{2}R_1\left[1 + \frac{2gh}{v^2\sin^2\theta}\right]^{1/2} - \frac{1}{2}R_1$$

$$= \frac{1}{2}R_1\left\{\left(1 + \frac{2gh}{v^2\sin^2\theta}\right)^{1/2} - 1\right\}$$

Hence $\dfrac{R_2 - R_1}{R_1} = \dfrac{1}{2}\left\{\left(1 + \dfrac{2gh}{v^2\sin^2\theta}\right)^{1/2} - 1\right\}$

Hence the Range is increased by $\frac{1}{2}\left\{\left[1 + \frac{2gh}{v^2\sin^2\theta}\right]^{1/2} - 1\right\}$ of its former value.

6(c) Evaluate the integral

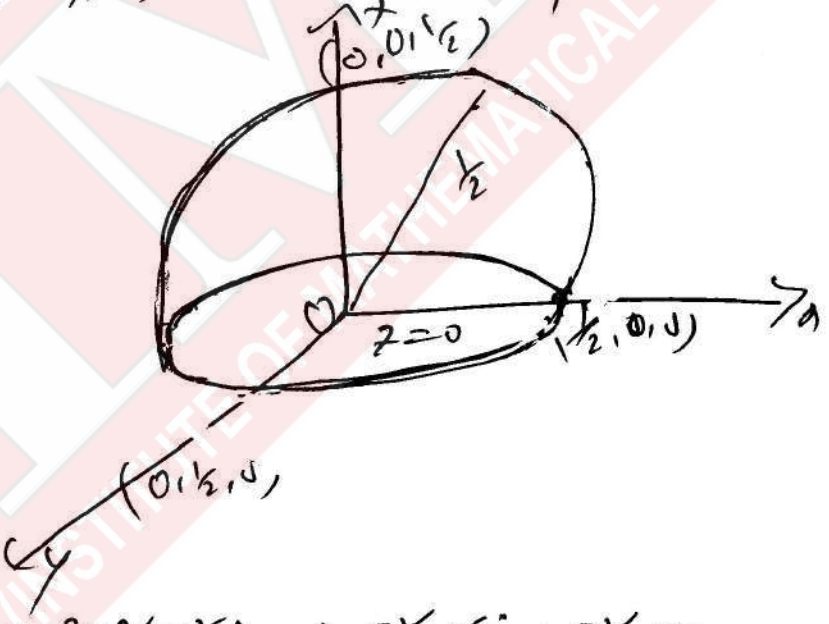
$$\iint_S [3y^2z^2 i + 4z^2x^2 j + z^2y^2 k]\cdot \hat{n}\, ds.$$

where 'S' is the upper part of the surface $4x^2+4y^2+4z^2=1$ above the plane $z=0$ and bounded by the xy-plane.

Hence verify Gauss-Divergence theorem;

Sol: Given surface 'S' is:

$$\phi(x,y,z) \equiv x^2+y^2+z^2-\tfrac{1}{4} (=0)$$



Let $\vec{F} = 3y^2z^2 i + 4z^2x^2 j + z^2y^2 k$.

we have

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = 2(xi+yj+zk)$$

we have

$$\vec{F}\cdot\hat{n} = 2(3xy^2z^2 + 4x^2z^2y + z^3y^2)$$

we have $ds = \frac{dxdy}{|\hat{n}\cdot k|} = \frac{dxdy}{2z}$

$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \iint_R \frac{z(3xy^2z^2+4x^2yz^2+y^2z^3)}{\cancel{2z}}\,dxdy$

$= \iint_R [3xy^2z+4x^2yz+y^2z^2)\,dxdy. \longrightarrow (1)$

corr. to $x^2+y^2=\frac{1}{4}$:

$x=r\cos\theta$; $y=\sin\theta$

$\therefore \boxed{dxdy = r\,dr\,d\theta}$ $\quad 0\le r\le\frac{1}{2}$; $0\le\theta\le 2\pi$

$\because x^2+y^2+z^2=\frac{1}{4} \Rightarrow z^2=\frac{1}{4}-x^2-y^2$

$\Rightarrow z=\sqrt{\frac{1}{4}-r^2}$

$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \int_{r=0}^{1/2}\int_{\theta=0}^{2\pi}\Big[3r^3\cos\theta\sin^2\theta\sqrt{\tfrac{1}{4}-r^2} + 4r^3\cos^2\theta\sin\theta\sqrt{\tfrac{1}{4}-r^2} + r^2\sin^2\theta\left(\tfrac{1}{4}-r^2\right)\Big]r\,dr\,d\theta$

$= \int_{r=0}^{1/2}\int_{\theta=0}^{2\pi}\Big[r^4\sqrt{\tfrac{1}{4}-r^2}\,(3\cos\theta\sin^2\theta+4\cos^2\theta\sin\theta) + \frac{r^3\sin^2\theta}{4} - r^5\sin^2\theta\Big]\,dr\,d\theta$

$= \int_{\theta=0}^{2\pi}\left[\sin^2\theta\left(\frac{1}{16\times 24}-\frac{1}{2^6\times 6}\right)\right]d\theta + \int_{r=0}^{1/2} r^4\sqrt{\tfrac{1}{4}-r^2}\left[\int_{\theta=0}^{2\pi}\left[3\cos\theta\sin^2\theta + 4\cos^2\theta\sin\theta\right]d\theta\right]$

$= \pi/768.$

2nd part:

we have $\nabla\cdot F = 2zy^2$.

$$\iiint \nabla\cdot F\, dx\,dy\,dz = \iiint (2zy^2)\,dx\,dy\,dz \quad \text{--- (2)}$$

corr. to $x^2+y^2+z^2 = \frac{1}{4}$

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$0 \le r \le \frac{1}{2}$

$0 \le \theta \le \frac{\pi}{2}$

$0 \le \phi \le 2\pi$

$$\therefore \iiint (\nabla\cdot F)\,dx\,dy\,dz = \int_{r=0}^{1/2}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} \left(2r\cos\theta\, r^2\sin^2\theta\cos^2\phi\right) r^2\sin\theta\, dr\,d\theta\,d\phi.$$

$$= 2\int_{r=0}^{1/2}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} r^5\cos\theta\sin^3\theta\cos^2\phi\, dr\,d\theta\,d\phi$$

$$= \frac{\pi}{768}.$$

$\therefore$ Gauss-divergence theorem

$$\iint_S \vec{F}\cdot\vec{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$

is verified

7. a (i) Find the solution of the differential equation

$$\frac{dy}{dx} = -\frac{2xy^3+2}{3x^2y^2+8e^{4y}}$$

Sol^n Rewriting the given ODE in form of $Mdx+Ndy=0$

as $(2xy^3+2)\,dx + (3x^2y^2+8e^{4y})\,dy = 0$

where $M = 2xy^3+2$ $\quad N = 3x^2y^2+8e^{4y}$

$\frac{\partial M}{\partial y} = 6xy^2$ $\quad \frac{\partial N}{\partial x} = 6xy^2$

Clearly $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ so It is exact.

So the general solution of exact ODE is of form $Mdx + Ndy = 0$ is given by

$$\int_{y=\text{const}} M\,dx + \int_{\substack{\text{term excluding}\\ x}} N\,dy = C$$

$$\Rightarrow \int_{y=\text{const}} (2xy^3+2)\,dx + \int 8e^{4y}\,dy = C$$

the term $3x^2y^2$ is excluded as it contains 'x^2'.

$$\Rightarrow 2y^3\frac{x^2}{2} + 2x + \frac{8}{4}e^{4y} = C$$

$$\Rightarrow x^2y^3 + 2x + 2e^{4y} = C \quad : C \text{ parameter}$$

is required general solution of given ODE.

7.a (ii) Reduce the equation $x^2p^2+y(2x+y)p+y^2=0$ to clairaut's form by the substitution $y=u$ and $xy=v$. Hence solve the equation and show that $y+4x=0$ is a singular solution of the differential equation.

Sol^n: The given equation is $x^2p^2+y(2x+y)p+y^2=0$

To convert it to clairaut's form it is given that the substitution to be used is

$y=u$ & $xy=v$.

$\Rightarrow dy=du \qquad ydx+xdy=dV$ or $x=\frac{V}{u}$

$$\text{So } dx = d\left(\frac{v}{u}\right) = \frac{udv-Vdu}{U^2}$$

$$\text{now } \frac{dy}{dx} = \frac{du \cdot U^2}{(udv-vdu)} = \frac{U^2}{u\frac{dv}{du}-V}$$

i.e $P = \dfrac{u^2}{UP-V}$ where $p=\frac{dy}{dx}$

$P=\frac{dv}{du}$.

now putting back the values of x,y & p in original equation we get

$$\frac{V^2}{u^2}\frac{u^4}{(uP-V)^2}+u\left(2\frac{V}{u}+u\right)\frac{u^2}{(UP-V)}+u^2=0$$

$$\Rightarrow \frac{V^2u^2}{(UP-V)^2}+\frac{2VU^2+u^4}{(UP-V)}+u^2=0$$

$\Rightarrow V^2u^2+(2VU^2+u^4)(uP-V)+u^2(UP-V)^2=0$

now squaring and opening terms we get

$$\cancel{V^2U^2}+2\cancel{V}U^3P+U^5P-2\cancel{V^2U^2}-VU^4+U^4P^2$$
$$+\cancel{U^2V^2}-2U^3\cancel{P}V=0$$

after cancellations of term we have

$U^5P - VU^4 + U^4P^2 = 0 \Rightarrow U^4(UP - V + P^2) = 0$

$\because u = y \neq 0$ so $UP - V + P^2 = 0$

or $V = UP + P^2$ —(i)

i.e (i) is reduced into clairaut's form

the general solution of (i) can be written directly as $V = uC + C^2$ —(ii)

where c is parameter.

To find singular solution we have to find p-discrimenent & c-discrimenent and for clairaut's form we know that p & c discrement are same.

so c-discremenent is given by $b^2 - 4ac = 0$ where the general solution is in Quadratic form. $c^2 + uc - v = 0$ Comparing it, we can write $U^2 - 4(1)(-V) = 0 \Rightarrow U^2 + V4 = 0$

Now converting it into x, y term we get

$y^2 + 4xy = 0 \Rightarrow y(y + 4x) = 0$

or $\boxed{y + 4x = 0}$ is a singular solution to the given equation.

And the general solution in x, y is written as $V = uc + c^2 \Rightarrow \boxed{xy = yc + c^2}$

Ans

7.(b) → A Solid Sphere is supported by a string fixed to a point on its rim and to a point on a smooth vertical wall with which the curved surface of the hemisphere is in contact. If θ, ϕ are the inclinations of the string and the plane base of the hemisphere to the vertical, prove that

$$\tan\phi = \frac{3}{8} + \tan\theta.$$

Solⁿ: O is a fixed point in the wall to which one end of the string has been attached.

Let 'l' be the length of the string AO and a be the radius of the hemisphere the centre of whose base is C.

The weight W of the hemisphere acts at its centre of gravity G which lies on the symmetrical radius CD and is such that $CG = \frac{3}{8}a$.

The hemisphere touches the wall at E.

we have $\angle OEC = 90°$ so that EC is horizontal.

The string AO makes an angle θ with the wall and the base BA of the hemisphere makes an angle ϕ with the wall.

The depth of G below $O = OF + AM + NG$
$$= l\cos\theta + a\cos\phi + \frac{3}{8}a\sin\phi$$

[Note that $\angle NCG = 90° - \angle ACM = 90° - (90° - \phi) = \phi$]

Give the system a small displacement in which θ changes to $\theta + \delta\theta$, ϕ changes to $\phi + \delta\phi$, the point O remains fixed, the length of the string AO does not change so that the work done by its tension is zero and the point G is slightly displaced the $\angle DEC$ remains 90°.

The only force that contributes to the equation of virtual work is the weight W of the hemisphere acting at G whose depth below the fixed point O has been found above.

The equation of virtual work is

$$W\delta\left(l\cos\theta + a\cos\phi + \frac{3}{8}a\sin\phi\right) = 0$$

$$\Rightarrow -l\sin\theta\,\delta\theta - a\sin\phi\,\delta\phi + \frac{3}{8}a\cos\phi\,\delta\phi = 0$$

$$\Rightarrow l\sin\theta\,\delta\theta = a\left(\frac{3}{8}\cos\phi - \sin\phi\right)\delta\phi. \quad (1)$$

$\therefore$ from the figure $EC = a$.

Also $EC = EM + MC = FA + MC$
$= l\sin\theta + a\sin\phi$

$a = l\sin\theta + a\sin\phi$

Differentiating, $0 = l\cos\theta\,\delta\theta + a\cos\phi\,\delta\phi$

$$\Rightarrow -l\cos\theta\,\delta\theta = a\cos\phi\,\delta\phi \quad (2)$$

Dividing (1) by (2), we get

$$-\tan\theta = \frac{3}{8} - \tan\phi$$

$$\tan\phi = \frac{3}{8} + \tan\theta.$$

7(C) If the tangent to a curve makes a constant angle θ with a fixed line, then prove that the ratio of radius of torsion to radius of curvature is proportional to $\tan\theta$. Further prove that if this ratio is constant, then the tangent makes a constant angle with a fixed direction.

Sol^n: We know that in calculus of space curves, we calculate the quantities curvature (K) and torsion (τ). Both have inverse-length as units, so their reciprocals $\frac{1}{K}$ and $\frac{1}{\tau}$ have units of lengths, and are called radius of curvature and radius of torsion.

Let e_1 be the unit vector parallel to the given fixed line so that as given $t \cdot e = \cos\alpha$ —— ①

Differentiating, we get

$\frac{dr}{ds} \cdot e = 0 \Rightarrow k n \cdot e = 0$ (Frenet's first)

$\therefore n \cdot e = 0$ — (2)

Hence, n is $\perp$ to e.

Thus, the vectors b, t, e are coplanar.

$\therefore b \cdot e = \pm \sin\alpha$ — (3)

Differentiating (2) and applying Frenet–Serret formula,

$\frac{dn}{ds} \cdot e = 0$

$-(kt + \tau b) \cdot e = 0$

$\therefore k\cos\alpha \pm \tau \sin\alpha = 0$

$\Rightarrow \frac{\cancel{1/\tau}}{1/k} = \frac{k}{\tau} = \tan\alpha$ = constant

from (1) & (3)

$\frac{k}{\tau} = \frac{1}{a}$, a is some scalar constant

$\frac{1}{k} = \frac{a}{\tau}$

i.e., $\sigma = a\rho$.

As $\frac{dr}{ds} = \frac{1}{\rho} n$ and $\frac{db}{ds} = \frac{1}{\sigma} n$

$\therefore \rho \frac{ds}{ds} = n = \sigma \frac{db}{ds}$

$$\frac{dt}{ds} = \frac{\sigma}{\rho}\frac{db}{ds} = a\frac{db}{ds}$$

Integrating, we get

$t = ab + c$,

where c is a constant vector.

Multiplying scalarly, with t,

we get $t \cdot t = ab \cdot t + c \cdot t$

$1 = 0 + c \cdot t$

i.e $c \cdot t = 1$

Hence, the tangent makes a constant angle with a fixed direction

8.(a) → Solve the following initial value problem by using Laplace transform technique

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y(t) = f(t),$$

$y(0) = 1,\ y'(0) = 0$ and $f(t)$ is a given function of t.

Sol^n: The given equation is $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y(t) = f(t)$

i.e $y'' - 4y' + 3y = f(t)$ —(i) and given initial

conditions as $y(0) = 1 \quad y'(0) = 0$

To solve using Laplace Transform Technique

So Taking L operator on given equation

$$L\{y'' - 4y' + 3y\} = L\{f(t)\}$$

$$\Rightarrow L\{y''\} - 4L\{y'\} + 3L\{y(t)\} = L\{f(t)\} \quad \text{— (ii)}$$

We know that $L\{y''\} = p^2 L\{y(t)\} - p\,y(0) - y'(0)$

$L\{y'\} = p\,L\{y(t)\} - y(0)$

So $L\{y''\} = p^2 . L\{y(t)\} - p - 0$

$L\{y'\} = p\,L\{y(t)\} - 1$

Now putting back the values in (ii)

$$p^2 L\{y(t)\} - p - 4p\,L\{y(t)\} + 4 + 3L\{y(t)\} = L\{f(t)\}$$

$$L\{y(t)\}\left[p^2 - 4p + 3\right] - p + 4 = L\{f(t)\}$$

$$\text{So } L\{y(t)\} = \frac{L\{f(t)\} + p - 4}{p^2 - 4p + 3}$$

$\because p^2 - 4p + 3 = (p-3)(p-1)$

$$\text{So } L\{y(t)\} = \frac{f(p)}{(p-3)(p-1)} + \frac{p-4}{(p-3)(p-1)}$$

where $L\{f(t)\} = f(p)$.

$\Rightarrow L^{-1}\{f(p)\} = f(t)$.

$$L\{y(t)\} = \frac{1}{2}\left[\frac{3}{p-1} - \frac{1}{p-3}\right] + \frac{1}{2} f(p)\left[\frac{1}{p-3} - \frac{1}{p-1}\right]$$

(on resolving into partial fractions)

$$y = \frac{3}{2} L^{-1}\left(\frac{1}{p-1}\right) - \frac{1}{2} L^{-1}\left(\frac{1}{p-3}\right) + \frac{1}{2} L^{-1}\left(f(p)\frac{1}{p-3}\right) - \frac{1}{2} L^{-1}\left\{f(p)\frac{1}{p-1}\right\}$$

$$= \frac{3}{2} e^{t} - \frac{1}{2} e^{3t} + \frac{1}{2} L^{-1}\{f(p)\, g(p)\} - \frac{1}{2} L^{-1}\{f(p)\, h(p)\} \quad (1)$$

where $g(p) = \frac{1}{p-3}$ and $h(p) = \frac{1}{p-1}$

so that $G(t) = L^{-1} g(p) = e^{3t}$ & $H(t) = L^{-1}\{h(p)\} = e^{t}$.

Now, by the convolution theorem, we have

$$L^{-1}\{f(p)\, g(p)\} = \int_0^t f(u)\, G(t-u)\, du$$

$$= \int_0^t f(u)\, e^{3(t-u)}\, du = e^{3t}\int_0^t f(u)\, e^{-3u}\, du \quad (2)$$

and

$$L^{-1}\{f(p)\, h(p)\} = \int_0^t f(u)\, H(t-u)\, du$$

$$= \int_0^t f(u)\, e^{t-u}\, du = e^{t}\int_0^t f(u)\, e^{-u}\, du \quad (3)$$

Using (2) & (3), (1) reduces to

$$y = \frac{1}{2}\left(3e^{t} - e^{3t}\right) + \frac{1}{2} e^{3t}\int_0^t f(u)\, e^{-3u}\, du - \frac{1}{2} e^{t}\int_0^t f(u)\, e^{-u}\, du.$$

8(b) A particle is projected from an apse at a distance $\sqrt{c}$ from the centre of force with a velocity $\sqrt{\frac{2\lambda}{3}}c^3$ and is moving with central acceleration $\lambda(r^5 - c^2 r)$. Find the path of motion of this particle. Will that be the curve $x^4+y^4=c^2$?

Sol^n: Here the central acceleration

$$P = \lambda(r^5 - c^2 r) = \lambda\left(\frac{1}{u^5} - \frac{c^2}{u}\right)$$

$\therefore$ The differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2}$$

$$= \frac{\lambda}{u^2}\left(\frac{1}{u^5} - \frac{c^2}{u}\right)$$

$$= \lambda\left(\frac{1}{u^7} - \frac{c^2}{u^3}\right)$$

Multiplying both sides by $2\left(\frac{du}{d\theta}\right)$ and then integrating, we have

$$v^2 = h^2\left(u^2 + \left(\frac{du}{d\theta}\right)^2\right) = \lambda\left[-\frac{1}{3u^6} + \frac{c^2}{u^2}\right] + A \quad (1)$$

where A is constant.

But initially, when $r = \sqrt{c}$ i.e. $u = \frac{1}{\sqrt{c}}$ and $\frac{du}{d\theta} = 0$ (at an apse) and

$$v = \sqrt{\frac{2\lambda}{3}}c^3$$

$\therefore$ from ①, we have

$$\left(\sqrt{\frac{2\lambda c^3}{3}}\right)^2 = h^2\left(\frac{1}{\sqrt{c}}\right)^2 = \lambda\left(-\frac{c^3}{3}+c^3\right)+A$$

$$\frac{2\lambda c^3}{3} = \frac{h^2}{c} = \lambda\frac{2c^3}{3}+A$$

$$\therefore \quad A=0 \text{ and } h^2=\frac{2c^4\lambda}{3}$$

Substituting the values of h^2 and A in ①, we have

$$\frac{2\lambda c^4}{3}\left[u^2+\left(\frac{du}{d\theta}\right)^2\right] = \lambda\left(-\frac{1}{3u^6}+\frac{c^2}{u^2}\right)$$

$$c^4\left[u^2+\left(\frac{du}{d\theta}\right)^2\right] = \frac{3}{2}\left(-\frac{1}{3u^6}+\frac{c^2}{u^2}\right)$$

$$\Rightarrow \quad c^4\left(\frac{du}{d\theta}\right)^2 = -\frac{1}{2u^6}+\frac{3c^2}{2u^2}-c^4u^2$$

$$= \frac{1}{u^6}\left[-\frac{1}{2}+\frac{3}{2}c^2u^4-c^4u^8\right]$$

$$= \frac{1}{u^6}\left[-\frac{1}{2}-\left(c^4u^8-\frac{3}{2}c^2u^4\right)\right]$$

$$= \frac{1}{u^6}\left[-\frac{1}{2}-\left(c^2u^4-\frac{3}{4}\right)^2+\frac{9}{16}\right]$$

$$= \frac{1}{u^6}\left[\frac{1}{16}-\left(c^2u^4-\frac{3}{4}\right)^2\right]$$

$$c^4\left(\frac{du}{d\theta}\right)^2 = \frac{1}{u^6}\left[\left(\frac{1}{4}\right)^2-\left(c^2u^4-\frac{3}{4}\right)^2\right]$$

$$\therefore \quad c^2u^3\frac{du}{d\theta} = \sqrt{\left(\frac{1}{4}\right)^2-\left(c^2u^4-\frac{3}{4}\right)^2}$$

$$d\theta = \frac{c^2 u^3\, du}{\sqrt{\left(\frac{1}{4}\right)^2 - \left(c^2u^4 - \frac{3}{4}\right)^2}}$$

Putting $c^2u^4 - \frac{3}{4} = z \Rightarrow 4c^2u^3 du = dz$

$c^2u^3 du = \frac{1}{4}dz$

$$\Rightarrow 4\,d\theta = \frac{dz}{\sqrt{\left(\frac{1}{4}\right)^2 - z^2}}$$

Integrating,

$$4\theta + B = \sin^{-1}\left(\frac{z}{1/4}\right) = \sin^{-1}(4z)$$

where B is constant.

$$4\theta + B = \sin^{-1}(4c^2u^4 - 3)$$

But initially when $u = \frac{1}{\sqrt{c}}$, $\theta = 0$

$\therefore B = \sin^{-1}(1) = \pi/2$

$$4\theta + \frac{\pi}{2} = \sin^{-1}(4c^2u^4 - 3)$$

$$\sin\left(4\theta + \frac{\pi}{2}\right) = 4c^2u^4 - 3$$

$$\cos 4\theta = 4c^2u^4 - 3$$

$$4c^2u^4 = 3 + \cos 4\theta$$

$$4\frac{c^2}{r^4} = 3 + \cos 4\theta$$

$$4c^2 = r^4\left[3 + (2\cos^2 2\theta - 1)\right]$$

$$= 2r^4\left[1 + \cos^2 2\theta\right]$$

$$= 2r^4\left[(\cos^2\theta + \sin^2\theta)^2 + (\cos^2\theta - \sin^2\theta)^2\right]$$

$$4c^2 = 4r^4\left[\cos^4\theta + \sin^4\theta\right]$$

$$c^2 = (r\cos\theta)^4 + (r\sin\theta)^4 = x^4 + y^4$$

which is the required equation of the path.

8.(c) For a scalar point function ϕ and vector point function $\vec{f}$, prove the identity

$$\nabla\cdot(\phi\vec{f}) = \nabla\phi\cdot\vec{f} + \phi(\nabla\cdot\vec{f}).$$

Also find the value of $\nabla\cdot\left(\frac{f(r)}{r}\vec{r}\right)$ and then verify stated identity.

Sol^n: We have

$$\text{curl}(\phi\vec{f}) = \nabla\cdot(\phi\vec{f})$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot(\phi\vec{f})$$

$$= \sum\left\{i\cdot\frac{\partial}{\partial x}(\phi\vec{f})\right\} = \sum i\cdot\left(\frac{\partial\phi}{\partial x}\vec{f} + \phi\frac{\partial\vec{f}}{\partial x}\right)\Big\}$$

$$= \sum\left\{i\cdot\left(\frac{\partial\phi}{\partial x}\vec{f}\right)\right\} + \sum\left\{i\cdot\left(\phi\frac{\partial\vec{f}}{\partial x}\right)\right\}$$

$$= \sum\left\{\left(\frac{\partial\phi}{\partial x}i\right)\vec{f}\right\} + \sum\phi\left(i\cdot\frac{\partial\vec{f}}{\partial x}\right)\Big\}$$

[Note that $a\cdot(mb) = (ma)\cdot b = m(a\cdot b)$]

$$= \left\{\sum\left(\frac{\partial\phi}{\partial x}i\right)\right\}\cdot\vec{f} + \phi\sum\left(i\cdot\frac{\partial\vec{f}}{\partial x}\right)$$

$$= \nabla\phi\cdot\vec{f} + \phi(\nabla\cdot\vec{f})$$

We have

$$\nabla\cdot\left(\frac{f(r)}{r}\vec{r}\right) = \nabla\left(\frac{f(r)}{r}\right)\cdot\vec{r} + \frac{f(r)}{r}(\nabla\cdot\vec{r})$$

$$= \left[\frac{r\nabla f(r) - f(r)\nabla r}{r^2}\right]\cdot\vec{r} + \frac{f(r)}{r}(3)$$

$$= \frac{r f'(r)\nabla r - f(r)\frac{\vec{r}}{r}}{r^2}\cdot\vec{r} + \frac{3f(r)}{r}$$

$$= \left[\frac{r f'(r)\frac{\vec{r}}{r} - f(r)\frac{\vec{r}}{r}}{r^2}\right]\cdot\vec{r} + 3\frac{f(r)}{r}$$

$$= \frac{\left[f'(r) - \frac{f(r)}{r}\right] r^2}{r^2} + 3\frac{f(r)}{r}$$

$$= f'(r) - \frac{f(r)}{r} + 3\frac{f(r)}{r}$$

$$= f'(r) + \frac{2f(r)}{r}$$

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