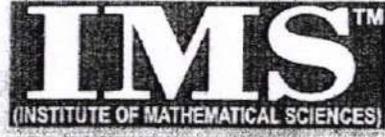


A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET



MAINS TEST SERIES-2021

(OCT. to DEC.-2021)

IAS/IFoS

MATHEMATICS

Under the guidance of K. Venkanna

FULL SYLLABUS (PAPER-II)

IAS(M)/21-NOV.-2021

 Test-16
 BATCH-I
 &
 Test-6
 BATCH-II

Time: 3 Hours

Maximum Marks: 250

INSTRUCTIONS

- This question paper-cum-answer booklet has 58 pages and has 34 PART/SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
- Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
- A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated.
- Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
- Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any **THREE** of the remaining questions selecting at least **ONE** question from each Section.
- The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
- Symbols/notations carry their usual meanings, unless otherwise indicated.
- All questions carry equal marks.
- All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
- All rough work should be done in the space provided and scored out finally.
- The candidate should respect the instructions given by the invigilator.
- The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any page from this booklet.

READ INSTRUCTIONS ON THE
LEFT SIDE OF THIS PAGE
CAREFULLY

Name Yadav - Suryabhan

Roll No. 0831222

Test Centre ORN
New Delhi
6-9 pm

Medium English

28th NOV 2021

Do not write your Roll Number or Name
anywhere else in this Question Paper-
cum-Answer Booklet.

I have read all the instructions and shall
abide by them

Signature of the Candidate

I have verified the information filled by the
candidate above

Signature of the invigilator

IMPORTANT NOTE:

Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED
1	(a)			03
	(b)			
	(c)			08
	(d)			08
	(e)			08
2	(a)			
	(b)			
	(c)			
	(d)			
3	(a)			11
	(b)			13
	(c)			08
	(d)			
4	(a)			04
	(b)			
	(c)			08
	(d)			13
5	(a)			08
	(b)			08
	(c)			01
	(d)			08
	(e)			08
6	(a)			06
	(b)			06
	(c)			11
	(d)			08
7	(a)			
	(b)			
	(c)			
	(d)			
8	(a)			
	(b)			
	(c)			
	(d)			
Total Marks				

27

32

25

33

31

148/250

SECTION - A

1. (a) If $G/Z(G)$ is cyclic, show that G is abelian, Where $Z(G)$ is centre of G .

[10]

$$Z(G) = \{x \in G; ax = xa\}$$

Given that $\frac{G}{Z(G)}$ is cyclic;

$$Zg_1, Zg_2 \in \frac{G}{Z}, \forall g_1, g_2 \in G$$

$$\frac{G}{Z} = \langle Zg_1 \rangle; \text{ since } \frac{G}{Z} \text{ is cyclic.}$$

~~let $x = Zg_1, y = Zg_2$~~

~~$xy = yx$ [as~~

As every cyclic group is abelian;

we have $(Zg_1) \cdot (Zg_2) = Zg_2 Zg_1$

or $Zg_1 g_2 = Zg_2 g_1$ { as Z is centre of group

$\Rightarrow g_1 g_2 (g_2 g_1)^{-1} \in Z$ { $g_1 Z = Zg_1$ }

{ from known theorem: if $x, y \in G$; and $\frac{G}{N}$ is abelian
 $xyx^{-1}y^{-1} \in N$; then G is abelian }

1. (c) A twice differentiable function f is such that $f(a) = f(b) = 0$ and $f(c) > 0$, for $a < c < b$. Prove that there is at least one value ξ between a and b for which $f''(\xi) < 0$. [10]

$$f(a) = f(b) = 0;$$

let $a < x_1 < c$; and $c < x_2 < b$.

In $a < x_1 < c$; we apply LMVT as the function is twice differentiable and thus is continuous

$$f'(x_1) = \frac{f(c) - f(a)}{c - a} \quad \text{--- ①}$$

also if $f(a) = f(b) = 0$; by roll's theorem we have $f'(x_3) = 0$ for some $x_3 \in [a, b]$

Now, similarly in $c < x_2 < b$;

applying LMVT, we have

$$f'(x_2) = \frac{f(b) - f(c)}{b - c};$$

Since function is twice differentiable, i.e. $f'(x)$ exists and is continuous. we apply LMVT in (x_1, x_2) ; s.t. $x_1 < \xi < x_2$ on $f'(x)$

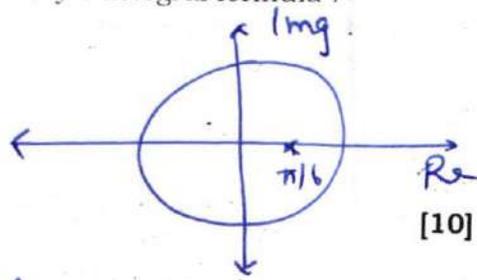
$$f''(\xi) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{-\frac{f(c) - f(b)}{b - c} - \frac{f(c) - f(a)}{c - a}}{x_2 - x_1} < 0$$

as $f(c) > 0$; & $x_2 - x_1 > 0$; $\therefore f''(\xi) < 0$

1. (d) Evaluate the following integrals by using Cauchy's integral formula :

(i) $\int_C \frac{(\sin z)^6}{\left(z - \frac{\pi}{6}\right)^3} dz$, where c is circle $|z| = 1$.

(ii) $\int_C \frac{e^{3z} dz}{z+i}$ if c is circle $|z + 1 + i| = 2$



Q (i) By Cauchy's integral

$$f(z) = I = \frac{2\pi i}{2!} \frac{d^2}{dz^2} (\sin z)^6 \Big|_{z=\pi/6}$$

$$= \pi i \frac{df}{dz} (6 \sin^5 z \cos z)$$

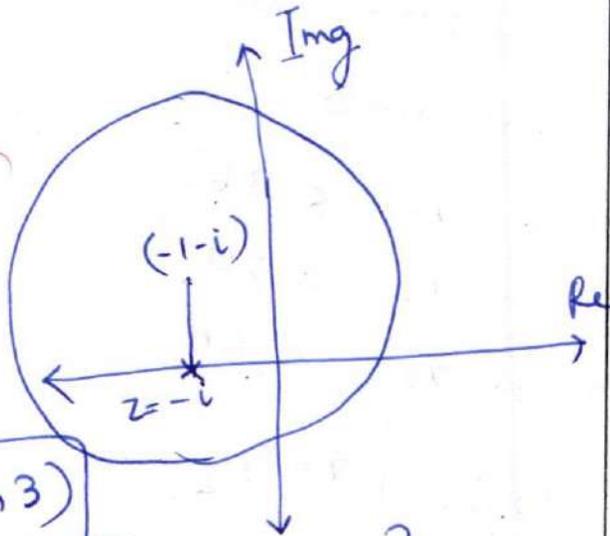
$$= (\pi i) [30 \sin^4 z \cos^2 z - 6 \sin^6 z] \Big|_{z=\pi/6}$$

$$= \pi i \left[\frac{30}{16} \times \frac{3}{4} - \frac{6}{64} \right]$$

$$= \underline{\underline{84\pi i / 64}}$$

(ii) $z = -i$, lies in
 $|z + 1 + i| = 2$

$$I = (2\pi i e^{-3i})$$



$$I = 2\pi i (\cos 3 + i \sin 3)$$

{ By Cauchy's integral theorem }

00

1. (e) A firm makes two types of furniture chairs and tables. The contribution for each product as calculated by the accounting department is Rs. 20/- per chair and Rs. 30/- per table. Both products are processed on three machines M_1 , M_2 , M_3 . The time required in hours by each product and total time available in hours per week on each machine are as follows :

Machine	Chairs	Table	Available Time
M_1	3	3	36
M_2	5	2	50
M_3	2	6	60

How should the manufacturer schedule his production in order to maximize contribution ? Solve graphically. [10]

Let x_1 be number of chair and
 x_2 be number of table.

Let Z denote the maximum profit
 function $Z_{\max} = 20x_1 + 30x_2$

from the given table, we form the
 below 3 constraints

$$3x_1 + 3x_2 \leq 36$$

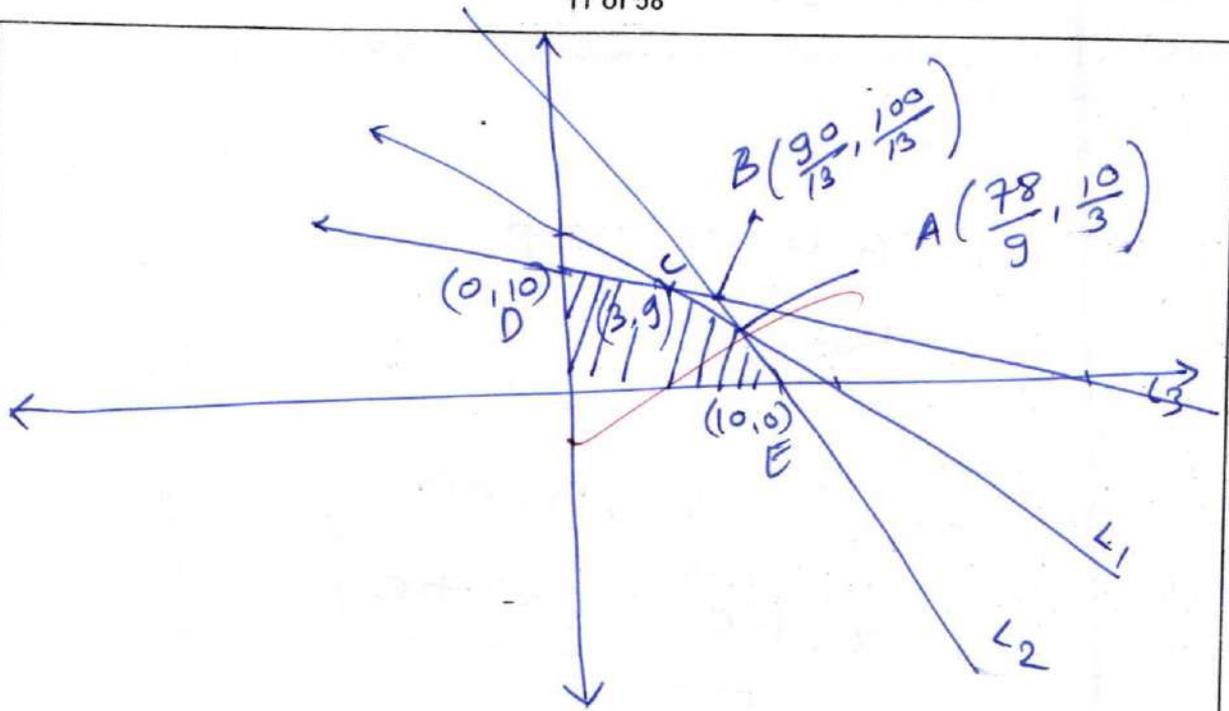
$$5x_1 + 2x_2 \leq 50$$

$$2x_1 + 6x_2 \leq 60$$

$$\text{or } x_1/12 + x_2/12 \leq 1 \rightarrow L_1$$

$$x_1/20 + x_2/25 \leq 1 \rightarrow L_2$$

$$x_1/30 + x_2/10 \leq 1 \rightarrow L_3$$



∴ the points of extremum

∴ $A = \left(\frac{78}{9}, \frac{10}{3}\right)$, $B^* = \left(\frac{90}{13}, \frac{100}{13}\right)$ lies outside convex set

$B = (3, 9)$, $D = (0, 10)$, $E = (10, 0)$

$$Z_A = \frac{20 \times 78 + 900}{9} = 2460/9$$

$$Z_C = 330, \quad Z_D = 300, \quad Z_E = 200$$

clearly Z_C is max;

thus $C = (3, 9)$ yields max profit

or $x_1 = 3$, $x_2 = 9$; i.e. 3 chair &

9 table will yield max profit.

2. (a) (i) Let x belong to a group. If $x^2 \neq e$ while $x^6 = e$, prove that $x^4 \neq e$ and $x^5 \neq e$. What can we say about the order of x ?

(ii) If $|a| = n$, show that $|a^t| = \frac{n}{\gcd(n,t)}$.

[18]

(i) $x \in G$, let G be a group
 $x^2 \neq e$, $x^6 = e$; then $x^{-2} \in G$.

as $x^6 \cdot x^{-2} \in G$;

$$x^4 = x^{-2} \in G; \Rightarrow o(x^4) = o(x^{-2})$$

$$\text{As if } x^2 \neq e \Rightarrow x^{-2} \neq e \Rightarrow \boxed{x^4 \neq e.}$$

$$x \cdot x = x^2 \neq e \Rightarrow \boxed{x \neq e}$$

$$\therefore x^{-1} \neq e.$$

$$x^6 \cdot x^{-1} = x^5 = x^{-1} \Rightarrow o(x^{-1}) = o(x^5)$$

$$\Rightarrow \boxed{x^5 \neq e.}$$

Also; as $x^2 \neq e$, $x^5 \neq e$,

$$\text{as } x^6 = e \Rightarrow o(x) \mid 6;$$

$$\text{i.e. } o(x) = 2, 3 \text{ or } 6.$$

$$\text{given } x^2 \neq e; \Rightarrow o(x) \neq 2;$$

$$\text{let's say } o(x) = 3 \Rightarrow \boxed{x^3 = e}$$

as we proved, $x \neq e$, $x^2 \neq e$,

i.e. 3 is minimum such possible;
 $\therefore \boxed{o(x) = 3}$ or multiple of 3.

3. (a) Let f be an isomorphism of a ring R onto a ring R' . Show that

(i) If R is an integral domain, then R' is also an integral domain.

(ii) If R is a field, then R' is also a field.

[18]

(i) $f: R \cong R'$; s.t. $f(0) = 0'$

Given that R is an integral domain;

let $r_1, r_2 \in R$; s.t. $r_1 r_2 = 0$; iff $r_1 = 0$ or $r_2 = 0$

$$\therefore f(r_1) \cdot f(r_2) = f(r_1 r_2) \quad \left\{ \begin{array}{l} \text{as } f \text{ is} \\ \text{isomorphism} \end{array} \right.$$

prove
unity
comas
as either r_1 or r_2 is 0, we have $f(r_1)$ or $f(r_2)$ as 0'.

{ where 0' is zero in R' }

$\therefore R'$ is also an integral domain

(ii) If R is a field, then R' is also a field.

$$f: R \rightarrow R'$$

i.e. every element in R has multiplicative inverse; to be proved.

As proved in above part, every R' is an integral domain, if R is an integral domain.

Also if every R is field, then its integral domain.

Thus R' is also integral domain.

$$f(x_1, x_2) = f(x_1)f(x_2) \quad \left\{ \begin{array}{l} \text{As } f \text{ is an} \\ \text{-omorphism} \end{array} \right.$$

$$x_1 = x_2^{-1} \Rightarrow$$

$x_1 x_2 = e$ where e is identity in R .
and $f(e) = e'$; e' is in R' .

$$\therefore f(e) = f(x_1) \cdot f(x_2)$$

$$\text{or } \boxed{f(x_1) \cdot f(x_2) = e'}$$

$$\textcircled{II} \text{ or } f(x_1) = [f(x_2)]^{-1};$$

thus every $f(x_i) \in R'$, has an inverse element

Hence R' is a field

3. (b) A function f is defined on $[0, 1]$ by

$$f(x) = x, \text{ if } x \text{ is rational} \\ = 1 - x, \text{ if } x \text{ is irrational.}$$

Show that f is not integrable on $[0, 1]$.

[15]

$$f(x) = x \quad ; \quad \text{if } x \text{ is rational} \\ = 1 - x \quad ; \quad \text{if } x \text{ is irrational}$$

Using Riemann integration

$$\text{let } P = \left\{ 0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{r-1}{n}, \frac{r}{n}, \dots, \frac{n}{n} = 1 \right\}$$

$$\text{and } \delta_r = \left[\frac{r-1}{n}, \frac{r}{n} \right];$$

$$\Delta = \frac{1}{n}$$

$$\text{for } U(P, f) = \sum M \cdot \Delta.$$

$$\text{as } f(x) = \begin{cases} x & \text{in } \frac{1}{2} < x < 1 \\ 1-x & \text{in } 0 < x < \frac{1}{2} \end{cases}$$

$$\text{we have } U(P, f) = \sum_{r=1}^n \int_0^{1/2} (1-x) dx + \int_{1/2}^1 x dx \\ = \frac{1}{2} - \frac{1}{8} + \frac{1}{2} - \frac{1}{8} = \frac{3}{4}$$

like for $L(P, f)$;

$$L = \int_0^{1/2} x dx + \int_{1/2}^1 (1-x) dx$$

$$= \left(\frac{1}{8}\right) + \left(x - \frac{x^2}{2}\right) \Big|_{1/2}^1$$

$$= \frac{1}{8} + 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{8} = 1$$

As $L(P, f) \neq U(P, f)$

Thus the given $f(x)$ is not

Rieman integrable.

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3. (c) Solve the following LPP by using Simplex method.

Maximize $z = 8x_1$, subject to the constraints :

$x_1 - x_2 \geq 0$, $2x_1 + 3x_2 \leq -6$ and x_1, x_2 are unrestricted.

[17]

as x_1, x_2 are unrestricted

$x_1 = x_1' - x_1''$, $x_2 = x_2' - x_2''$; $x_1', x_1'' \geq 0$, $x_2', x_2'' \geq 0$

$Z_{max} = 8x_1' - 8x_1''$; rewriting eqⁿ

$x_1' - x_1'' - x_2' + x_2'' + s_1 = 0$

$-2x_1' + 2x_1'' - 3x_2' + 3x_2'' - s_2 + A_1 = 6$

$Z^*_{max} = -0s_1 - 0s_2 - A_1$

CB	Basis	x_1'	x_1''	x_2'	x_2''	s_1	A	s_2	b/a
0	s_1	1	-1	-1	1	1	0	0	0
-1	A_1	-2	2	-3	3	0	1	-1	6
$C_j - Z_j$		-2	2	-3	+3	0	0	-1	

thus x_2'' is incoming variable and s_1 is outgoing variable. (as $\frac{b}{a} = 0$ for s_1)

$\min(0, 6/3 = 2)$

CB	Basis	x_1'	x_1''	x_2'	x_2''	s_1	A	s_2	b
0	x_2''	1	-1	-1	1	1	0	0	0
-1	A_1	-5	5	0	0	-3	1	-1	6
$C_j - Z_j$		-5	5	0	0	-3	0	-1	

where $Z_j = \sum C_j a_{ij}$

x_1'' is incoming variable and A_1 is incoming variable. (as $a_{12} < 0$)

C_B	Basis	x_1'	x_1''	x_2'	x_2''	s_1	s_2	b
0	x_2''	0	0	-1	1	2/5	-1/5	6/5
0	x_1''	-1	1	0	0	-3/5	-1/5	6/5
$G-Z_j$		0	0	0	0	0	0	0

Now we proceed with phase II of simplex method.

C_B	Basis	x_1	x_1''	x_2'	x_2''	s_1	s_2	b
0	x_1	0	0	+8	-8	0	0	0
-8	x_2''	0	0	-1	1	2/5	-1/5	6/5
0	x_1''	-1	1	0	0	-3/5	-1/5	6/5
$(C_j - Z_j)$		0	0	0	0	+	-	

clearly x_2'' is outgoing and s_1 is incoming in basis

OB

Solve further

4. (a) Find the g.c.d. of $11 + 7i$ and $18 - i$ in $\mathbb{Z}[i]$.

[12]

$$\cancel{11+7i} \frac{18-i}{11+7i} = \frac{(18-i)(11-7i)}{11^2+7^2} = \frac{(198-7) - i(137)}{170}$$

$$\frac{(18-i)}{11+7i} = \frac{191}{170} - \frac{137i}{170}$$

$$= (1-i) + \left(\frac{21}{170} + \frac{33i}{170} \right)$$

$$\left| \frac{21}{170} + \frac{33i}{170} \right| < |1-i|$$

$$\therefore (18-i) = (11+7i)(1-i) + \frac{(11+7i)(21+33i)}{170}$$

$$= (11+7i)(1-i) + 3i$$

$$\frac{3i}{1-i} = \frac{3i(1+i)}{2} = \frac{3i}{2} - \frac{3}{2}$$

$$\text{or } = \frac{(-1+i)}{2} + \frac{(-1+i)}{2}$$

$$(3i) = (1-i)(-1+i) + \frac{(1-i)(-1+i)}{2}$$

$$\underline{d(0)} < d(-1+i); \text{ again}$$

we have G.C.D as $\underline{(-1+i)}$

4. (c) (i) If $u - v = (x - y)(x^2 + 4xy + y^2)$ and $f(z) = u + iv$ is an analytic function of $z = x + iy$, find $f(z)$ in terms of z .
 (ii) The function $f(z)$ has a double pole at $z = 0$ with residue 2, a simple pole at $z = 1$ with residue 2, is analytic at all other finite points of the plane and is bounded as $|z| \rightarrow \infty$. If $f(2) = 5$, $f(-1) = 2$, find $f(z)$.

[12]

$$(i) \quad f(z) = u + iv; \quad i f(z) = u i - v$$

$$(1+i)f(z) = u - v + i(u + v) = U + iV$$

$$\frac{\partial U}{\partial x} = \frac{\partial (u - v)}{\partial x} = \frac{\partial (x - y)(x^2 + 4xy + y^2)}{\partial x}$$

$$\phi_1(z_0) = x^2 + 4xy + y^2 + (x - y)(2x + 4y)$$

$$\frac{\partial U}{\partial y} = -(x^2 + 4yx + y^2) + (x - y)(2y + 4x)$$

$$\left. \frac{\partial U}{\partial x} \right|_{(z,0)} = 3z^2; \quad \left. \frac{\partial U}{\partial y} \right|_{(z,0)} = 3z^2 = \phi_2(z_0)$$

$$\therefore f(z)(1+i) = \int [\phi_1(z,0) - i\phi_2(z,0)] dz$$

$$f(z) = \int \frac{3z^2(1-i)}{1+i} dz$$

$$f(z) = \frac{z^3}{2} (1 + (-i)^2 - 2i) + C$$

$$\text{or } \boxed{f(z) = -iz^3 + C}$$

(ii) By Laurent's expansion of $f(z)$

$$f(z) = \sum a_n z^n + \frac{2}{z} + \frac{2}{z-1}$$

further ; as $z \rightarrow 0$; $|f(z)|$ is bounded
 means; $a_1 = a_2 = \dots = a_n = 0$; $a_0 \neq 0$.

or

$$f(z) = a + \frac{2}{z} + \frac{b}{z^2} + \frac{2}{z-1}$$

given that $f(2) = 5$, $f(-1) = 2$

$$5 = a + 1 + \frac{b}{4} + 7 \quad \text{--- } \textcircled{1}$$

$$a + \frac{b}{4} = 3 \quad \text{--- } \textcircled{2}$$

$$2 = a - 2 + b - 1$$

$$a + b = 5$$

$$b = 8/3$$

$$a = 7/3$$

$$a = 1$$

$$b = 4$$

$$\therefore f(z) = \frac{7}{3} + \frac{2}{z} + \frac{8/3}{z^2} + \frac{2}{z-1}$$

$\textcircled{08}$

4. (d) Five salesmen are to be assigned to five territories. Based on the past performance, the following table shows the annual sales (in rupees lakhs) that can be generated by each salesman in each territory. find the optimum assignment.

Salesman	T_1	T_2	T_3	T_4	T_5
S_1	26	14	10	12	9
S_2	31	27	30	14	16
S_3	15	18	16	25	30
S_4	17	12	21	30	25
S_5	20	19	25	16	10

clearly this is maximisation problem. [15]
 Thus ($\times -1$) each element and then
 proceeding to minimise cost.

-26	-14	-10	-12	-9
-31	-27	-30	-14	-16
-15	-18	-16	-25	-30
-17	-12	-21	-30	-25
-20	-19	-25	-16	-10

5	17	21	19	22
0	4	1	17	15
16	13	15	6	1
14	18	10	1	6
11	12	6	15	21

0	12	16	14	17
0	4	1	17	15
15	12	14	5	0
13	17	9	0	5
5	6	0	9	15

Subtracting by
 row min and
 column minima
 to have 0 in
 every row &
 column.

0	8	16	14	17
0	0	1	17	15
15	8	14	5	0
13	13	9	0	9
5	2	0	9	15

$$S_1 \rightarrow T_1$$

$$S_2 \rightarrow T_2$$

$$S_3 \rightarrow T_5$$

$$S_4 \rightarrow T_4$$

$$S_5 \rightarrow T_5$$

Also optimum cost = $26 + 27 + 30 + 30 + 25$

$$= \underline{\underline{138}} \text{ lakh rupees}$$

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SECTION - B

5. (a) Solve $(D^2 - 4D')z = (4x/y^2) - (y/x^2)$.

[10]

Auxiliary function is
 $m^2 - 4 = 0 \Rightarrow m = \pm 2;$

\therefore C.F is $\phi_1(y+2x) + \phi_2(y-2x)$

$$P.I = \frac{1}{D^2 - 4D'} \left[\left(\frac{4x}{y^2} \right) - \left(\frac{y}{x^2} \right) \right]$$

$$= \frac{1}{-4D'^2 \left(1 - \frac{D^2}{4D'^2} \right)} \left\{ \frac{4x}{y^2} \right\} - \frac{1}{D^2 \left(1 - \frac{4D'^2}{D^2} \right)} \left(\frac{y}{x^2} \right)$$

$$= \frac{1}{-4D'^2} \left(1 + \frac{D^2}{4D'^2} + \dots \right) \left(\frac{4x}{y^2} \right) - \frac{1}{D^2} \left(1 + \frac{4D'^2}{D^2} + \dots \right) \frac{y}{x^2}$$

$$= -\frac{1}{4D'^2} \left(\frac{4x}{y^2} \right) - \frac{1}{D^2} \left(\frac{y}{x^2} \right)$$

$$= \frac{08}{08} + \underline{x \log y} + \underline{y \log x}$$

complete solution is

$$z = \phi_1(y+2x) + \phi_2(y-2x) + x \log y + y \log x$$

5. (b) Reduce $y^2(\partial^2 z / \partial x^2) + x^2(\partial^2 z / \partial y^2) = 0$ to canonical form

[10]

Comparing with $R\lambda^2 + S\lambda + T = 0$; we have $R = y^2$, $S = 0$, $T = x^2$.

$$\therefore \lambda^2 y^2 + x^2 = 0;$$

$$\text{or } \lambda = \pm \frac{x}{y} i$$

$$\frac{dy}{dx} + \lambda = 0 \Rightarrow \frac{dy}{dx} + \frac{x}{y} i = 0$$

$$\text{or } K = \frac{ix^2}{2} \pm \frac{y^2}{2} \rightarrow \text{constant}$$

$$\text{let } \alpha = \frac{y^2}{2}; \quad \beta = \frac{x^2}{2}$$

$$K = \pm d + \beta i$$

$$\left(\frac{\partial z}{\partial x}\right) = \frac{\partial z}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta} \cdot \frac{\partial \beta}{\partial x} = x \frac{\partial z}{\partial \beta}$$

$$\frac{\partial^2 z}{\partial x^2} = x^2 \frac{\partial^2 z}{\partial \beta^2} + \frac{\partial z}{\partial \beta}$$

$$\text{likewise } \frac{\partial^2 z}{\partial y^2} = y^2 \frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial z}{\partial \alpha}$$

Putting

$$x^2 y^2 \frac{\partial^2 z}{\partial \beta^2} + y^2 \frac{\partial z}{\partial \beta} + x^2 y^2 \frac{\partial^2 z}{\partial \alpha^2} + x^2 \frac{\partial z}{\partial \alpha} = 0$$

$$\frac{\partial^2 z}{\partial \beta^2} + \frac{1}{2\beta} \frac{\partial z}{\partial \beta} + \frac{\partial^2 z}{\partial \alpha^2} + \frac{1}{2\alpha} \frac{\partial z}{\partial \alpha} = 0$$

5. (c) The equation $2e^{-x} = \frac{1}{x+2} + \frac{1}{x+1}$ has two roots greater than -1 . Calculate these roots correct to five decimal places.

$$f(x) = 2e^{-x} - \frac{1}{(x+2)} - \frac{1}{(x+1)} \quad [10]$$

$$f(-1) = 4.60, \quad f(0) = 0.5,$$

$$f(1) = 0.0975,$$

Using Newton's Raphson method

$$x_2 = \frac{f(x_1)}{f'(x_1)} \quad \text{let } x_1 = 0$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$f'(x) = -2e^{-x} + \frac{1}{(x+2)^2} + \frac{1}{(x+1)^2}$$

(01)

5. (d) Convert the following :

(i) $(41.6875)_{10}$ to binary number

(ii) $(101101)_2$ to decimal number

(iii) $(AF63)_{16}$ to decimal number

(iv) $(101111011111)_2$ to hexadecimal number

[10]

$$\textcircled{1} (41.6875)_{10} = (101001.1011)_2$$

$$41 = 2^5 \times 1 + 2^3 \times 1 + 2^0 \times 1$$

$$0.6875 \times 2 = 1.375 \quad \textcircled{1}$$

$$0.375 \times 2 = 0.750 \quad \textcircled{0}$$

$$0.750 \times 2 = 1.50 \quad \textcircled{1}$$

$$0.50 \times 2 = 1.00 \quad \textcircled{1}$$

$$\textcircled{2} (101101)_2 = 1 \times 2^5 + 0 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$$

$$= (45)_{10}$$

$$\textcircled{3} (AF63)_{16} = \textcircled{1010111101000011}_2$$

$$= 2^{15} + 2^{13} + 2^{12} + 2^{10} + 2^8 + 2^9 + 2^6 + 2^5 + 2^1 + 2^0$$

$$= (44899)_{10}$$

$$\textcircled{4} (101111011111)_2 = \textcircled{BDF}_{16}$$

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5. (e) A velocity field is given by $\mathbf{q} = -x\mathbf{i} + (y+t)\mathbf{j}$. Find the stream function and the stream lines for this field at $t = 2$. [10]

$$\vec{q} = -x\mathbf{i} + (y+t)\mathbf{j}$$

stream function is ~~the~~ $\frac{dx}{u} = \frac{dy}{v}$
line

$$\frac{dx}{-x} = \frac{dy - dz}{y+t} \Rightarrow \frac{dx}{-x} = \frac{dy}{y+2}$$

$(y+2)x = C$ is the required stream line
and $z = \text{constant}$

for stream function:

$$u = \frac{-\partial\psi}{\partial y} \quad ; \quad v = \frac{\partial\psi}{\partial x}$$

$$\text{or } \psi = \int \frac{\partial\psi}{\partial x} dx + \frac{\partial\psi}{\partial y} dy$$

$$= \int v dx - u dy$$

$$\text{(OB)} = \int (y+t) dx + x dy$$

$\psi = tx + xy + C_2$ is the required stream function.

6. (a) Form partial differential equation by eliminating arbitrary functions f and g from $z = f(x^2 - y) + g(x^2 + y)$.

$$\frac{\partial z}{\partial x} = 2x f'(x^2 - y) + 2x (g'(x^2 + y)) \quad [08]$$

$$\frac{1}{2x} \frac{\partial z}{\partial x} = f' + g' \quad \text{--- (1)}$$

$$\frac{\partial z}{\partial y} = -f' + g' \quad \text{--- (2)}$$

$$\frac{\partial^2 z}{\partial y^2} = f'' + g'' \quad \text{--- (3)}$$

diff (1) once again w.r.t (2)

$$-\frac{1}{2x^2} \left(\frac{\partial z}{\partial x} \right) + \frac{1}{2x} \frac{\partial^2 z}{\partial x^2} = f'' + g''$$

$$\frac{\partial^2 z}{\partial y^2} = f'' + g''$$

$$\text{or } \boxed{-\frac{1}{2x^2} \frac{\partial z}{\partial x} + \frac{1}{2x} \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0}$$

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6. (b) Find a complete integral of $16p^2z^2 + 9q^2z^2 + 4z^2 - 4 = 0$.

using Charpit's method

[08]

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-f_z/p}$$

$$\frac{dp}{3/2pz^2 + p(32p^2z + 18q^2 + 8z)} = \frac{dq}{-f_z/p}$$

$$\frac{dp}{pf_z} = \frac{dq}{qf_z} \Rightarrow \boxed{p = aq}$$

$$\therefore (16a^2z^2 + 9z^2)q^2 = (4z^2 - 4)$$

$$\text{or } q^2 = \pm \frac{\sqrt{4(1-z^2)}}{\sqrt{16a^2z^2 + 9z^2}}$$

$$\boxed{dz = p dx + q dy}$$

$$dz = (adx + dy) q$$

$$\text{or } \int adx + dy = \int \frac{(\sqrt{16a^2 + 9})z}{2\sqrt{1-z^2}} dz$$

$$\int (adx + dy) = \frac{-\sqrt{16a^2 + 9} \cdot \sqrt{1-z^2}}{2} + b$$

$$\boxed{(ax + by + b)^2 = \frac{(16a^2 + 9)(1-z^2)}{4}}$$

6. (c) Find the characteristics of the equation $pq = xy$ and determine the integral surface which passes through the curve $z = x, y = 0$. [16]

$$\frac{dx}{dt} = \frac{df}{dp}$$

where

$$f = pq - xy$$

$$\frac{dx}{dt} = q, \quad \frac{dy}{dt} = p$$

$$\frac{dz}{dt} = 2pq$$

$$\frac{dp}{dt} = -y, \quad \frac{dq}{dt} = +x$$

also; the curve passes through $z = x, y = 0$; we have $x = z = \lambda_0, y = 0$
or $p_0 q_0 = 0$ — (1)

$$f'_3(\lambda) = p_0 f'_1(\lambda_0) + q_0 f'_2(\lambda_0)$$

$$1 = p_0 + q_0(0) \quad \text{or} \quad p_0 = 1$$

$$q_0 = 0$$

$$\frac{dx}{dt} = \frac{dq}{dt} \Rightarrow q = x$$

$$p + q + x + y = \frac{d}{dt}(p + q + x + y)$$

$$p + q + x + y = e^t \cdot C; \text{ as}$$

$$x_0 = \lambda_0, y = 0, p_0 = 1, q_0 = 0, \text{ at } t = 0;$$

$$C = 1 + \lambda_0$$

or

$$p + q + (x + y) = (\lambda_0 + 1)e^t \quad (1)$$

$$(p+q) - (x+y) = -\frac{d}{dt} (p+q - (x+y))$$

$$(p+q) - (x+y) = c'e^{-t};$$

where $c' = 1-\lambda$.

$$(p+q) - (x+y) = (1-\lambda)e^{-t} \quad \text{--- (2)}$$

$$\frac{d}{dt} (p-q + x-y) = y-x + q-p = - (p-q + (x-y))$$

$$p-q + (x-y) = c^2 e^{-t}$$

or $p-q + (x-y) = (1+\lambda)e^{-t} \quad \text{--- (3)}$

$$\frac{d}{dt} (p-q - x+y) = y-x - q+p = e^t c \quad \text{OR}$$

$$(p-q) - (x-y) = e^t c$$

$$(p-q) - (x-y) = e^t (1-\lambda) \quad \text{--- (4)}$$

Adding (1) & (3); q (3) & (4)

$$p+q = (2+\lambda)e^t + (1-\lambda)e^{-t}$$

$$p-q = e^t(1-\lambda) + e^{-t}(1+\lambda)$$

$$p = \frac{1}{2}(e^t + e^{-t}); \quad q = \frac{\lambda}{2}(e^t - e^{-t})$$

$$x+y = (2+\lambda)e^t - (1-\lambda)e^{-t}$$

$$x-y = e^t(1+\lambda) - e^{-t}(1-\lambda)$$

$$x = \lambda(e^t + e^{-t}); \quad y = \frac{1}{2}(e^t - e^{-t})$$

Check
Calculated
-1 or

$$\frac{dz}{dt} = 2\lambda (e^{2t} - e^{-2t}) \quad \text{OR}$$

$$z = 2\lambda \left(\frac{e^{2t}}{2} + \frac{e^{-2t}}{-2} \right) + C$$

$$z = 2\lambda \left(\frac{e^{2t}}{2} + \frac{e^{-2t}}{-2} \right) - 3\lambda$$

6. (d) A thin rectangular homogeneous thermally conducting plate lies in the xy -plane defined by $0 \leq x \leq a$, $0 \leq y \leq b$. The edge $y = 0$ is held at the temperature $Tx(x - a)$, where T is a constant, while the remaining edges are held at 0° . The other faces are insulated and no internal sources and sinks are present. Find the steady state temperature inside the plate. [18]

the heat equation is Laplacian for 2-D flow in steady state temperature condition

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u(x, y) \text{ represents the heat flow.}$$

given boundary conditions are:

$$(i) \quad u(0, y) = u(a, y) = 0$$

$$(ii) \quad u(x, 0) = f(x); \quad u(x, b) = 0.$$

~~$$\frac{\partial^2 u}{\partial x^2} = \lambda^2$$~~

Using method of separation of variables

to solve this problem.

let $u = X \cdot Y$; we have

$$X \frac{d^2 X}{dx^2} + Y \frac{d^2 Y}{dy^2} = 0$$

$$\text{let } X \frac{d^2 X}{dx^2} = -\lambda^2$$

we here cannot take $+\lambda^2$ or 0 , as it gives impossible solution that doesn't satisfy boundary condition.

$$X = A \sin \lambda x + B \cos \lambda x;$$

$$Y = A_1 e^{\lambda y} + B_1 e^{-\lambda y}$$

$$\text{or } u = XY = \sum (A \sin \lambda x + B \cos \lambda x) (A_1 e^{\lambda y} + B_1 e^{-\lambda y})$$

at $x=0, x=a; u=0 \Rightarrow B=0$ and

$$\lambda a = n\pi$$

Now; $u(x,b)=0$; where $A' = AA_1$
 $B' = AB_1$ (3)

or $A'e^{\lambda b} + B'e^{-\lambda b} = 0$ (3)

for 4th B.C; we have
 $f(x) = (A'e^{\lambda(0)} + B'e^{-\lambda(0)}) \sin(\lambda x)$

or $f(x) = (A'+B') \sin \lambda x$

or $(A'+B') = \frac{2}{a} \int_0^a f(x) \sin \lambda x dx$

$(A'+B') = \frac{2T}{a} \int_0^a (x^2 \sin \lambda x - ax \sin \lambda x) dx$

$(A'+B') = \frac{2T}{a} \left\{ \frac{-x^2 \cos \lambda x}{\lambda} + \frac{2x \sin \lambda x}{\lambda^2} \right.$

(3)

$\left. + \frac{2 \cos \lambda x}{\lambda^3} \right|_0^a$

$-a \left\{ \frac{-x \cos \lambda x}{\lambda} + \frac{\sin \lambda x}{\lambda^2} \right\} \Big|_0^a$

$A'+B' = \frac{2T}{a} \left(\frac{-a^2 [\cos(n\pi) - 1]}{\lambda} + \frac{2 \sin(n\pi)}{\lambda^2} + \frac{2 [\cos(n\pi) - 1]}{\lambda^3} \right)$

$+ a^2 \frac{\cos(n\pi)}{\lambda} - \frac{a}{\lambda^3} \sin(n\pi) \Big|_0^a$ (4)

Final solⁿ is; A', B' can be calculated from (3) & (4)

$f(x) = \sum_{n=0}^{\infty} (A'e^{\lambda y} + B'e^{-\lambda y}) \sin \lambda x$; $\lambda = n\pi/a$

Refer
 key