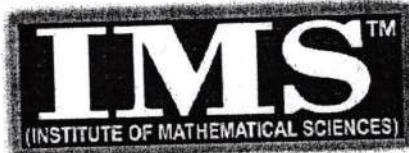


## A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET

**MAINS TEST SERIES-2020**

(OCT. TO JAN.-2020-21)

IAS/IFoS  
**MATHEMATICS**  
Under the guidance of K. Venkanna

FULL SYLLABUS (PAPER-I)

DATE : 13-DEC-2020

Common Test  
Test-17 for Batch-I  
&  
Test-9 for Batch-II

**Time: 3 Hours****Maximum Marks: 250****READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY**Name *Surejabhan Yadav*Roll No. *0506355*Test Centre *[Blank]*Medium *[Blank]*

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

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I have read all the instructions and shall abide by them

Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the invigilator

**IMPORTANT NOTE:**

Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

# INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED
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<b>Total Marks</b>				

## SECTION - A

1. (a) If  $A$  is both real symmetric and orthogonal, prove that all its eigenvalues are  $+1$  or  $-1$ . [10]

$$A = A^T \quad (\text{since } A \text{ is symmetric})$$

$$\cancel{AA^T = I} \quad (\text{since } A \text{ is orthogonal})$$

$$\cancel{A^2 = I} \quad \text{How??}$$

$$\text{or } |A| |A| = I$$

$$\text{or } |A|^2 = I$$

$$|A| = \pm 1$$

$$\{\det(AB) = \det A \cdot \det B\}$$

$$AX = \lambda X \quad ; \quad \text{and} \quad A^T X = \lambda X$$

$$A^T A^T X = \lambda A^T X \quad \{ \text{multiply by } A^T \}$$

$$IX = \lambda(\lambda X)$$

$$\text{or } (\lambda^2 - 1)X = 0$$

$$\lambda = \pm 1$$

here all eigen values are either  $+1$

or  $-1$

Q8 ✓

1. (c) If  $f(x,y) = \begin{cases} \frac{x^3+y^3}{x-y}, & x \neq y \\ 0, & x=y \end{cases}$ , show that the function is discontinuous at the origin

but possesses partial derivatives  $f_x$  and  $f_y$  at every point, including the origin. [10]

$$\lim_{x,y \rightarrow 0} f(x,y) = \lim_{\substack{y \rightarrow x - mx^3 \\ x \rightarrow 0}} \frac{x^3+y^3}{x-y} = \frac{x^3 + x^3(1-mx^2)^2}{x-x+mx^3} = \frac{1+(1-mx^2)^3}{m \cancel{x^2}}$$

$$= 2/m ; \neq 0 \text{ as } x \rightarrow 0$$

and thus this value varies;

hence its discontinuous at  $(0,0)$ .

$$f_x = \frac{d}{dx} (f(x,y)) = \frac{f(h,0) - f(0,0)}{h} = \underset{h \rightarrow 0}{\lim} \frac{h^3 + 0}{h \cdot 0} = 0$$

$$f_x = \lim_{h \rightarrow 0} \frac{f(h,0) - f(0,0)}{h}$$

$$f_y = \lim_{k \rightarrow 0} \frac{f(0,k) - f(0,0)}{k} = 0$$

hence partial derivative  $= 0$  at  $(0,0)$ .

$$\underline{f_x = f_y = 0}$$

$$f_x = \frac{d}{dx} f(x,y) = \frac{d}{dx} \left( \frac{x^3+y^3}{x-y} \right) = \frac{(x-y)(3x^2) + (x^3+y^3)}{(x-y)^2}$$

$$f_x = \frac{4x^3 + y^3 - 3x^2y}{(x-y)^2}; x \neq y;$$

$$\text{and } f_y = \frac{d}{dy} f(x, y) = \frac{(xy)(3y^2 + (xy))(x^3 + y^3 - 1)}{(x-y)^2}$$

$$= \frac{3xy^2 - x^3 - 4y^3}{(x-y)^2}; x \neq y$$

hence partial derivative exists at  $(0, 0)$  ;  
and other point; but function is discontinuous

Q9'

1. (d) If  $V = \log_e \sin \left\{ \frac{\pi(2x^2 + y^2 + xz)^{1/2}}{2(x^2 + xy + 2yz + z^2)^{1/3}} \right\}$ , find the value of

$$x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y} + z \frac{\partial V}{\partial z} \text{ when } x = 0, y = 1, z = 2.$$

$$f(V) = \sin^+ e^V = \frac{\pi (2x^2 + y^2 + xz)^{1/2}}{2(x^2 + xy + 2yz + z^2)^{1/3}}; \text{ is of order } \left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{16}$$

from Euler's theorem :-

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} - nf; \text{ here } n = \frac{1}{16}, \quad n = \frac{1}{13}$$

value of  $f$ ; at  $x = 0, y = 1, z = 2$

$$f = \frac{\pi}{2} \frac{(0+1+0)^{1/2}}{(0+0+4+4)^{1/3}} = \frac{\pi}{4}$$

$$\frac{\partial f}{\partial x} = \frac{1}{\sqrt{1-e^{2V}}} e^V \cdot \frac{\partial V}{\partial x}$$

value of  $e^V$  at  $x=0, y=1, z=2$

$$e^V = \sin \left[ \frac{(\pi)(2x^2+yz+z^2)^{1/2}}{2(x^2+xy+y^2+z^2)^{1/2}} \right] = \sin \left( \frac{\pi}{4} \right) = \frac{1}{\sqrt{2}}$$

$$\frac{\partial f}{\partial x} = \frac{1}{\sqrt{2}} \left( 1 - \frac{1}{2} \right) \frac{\partial V}{\partial x} = \frac{\partial V}{\partial x} \quad ? \text{ Likewise}$$

$\frac{\partial V}{\partial y} = \frac{\partial f}{\partial y}, \frac{\partial V}{\partial z} = \frac{\partial f}{\partial z}$ , hence from ①

$$\frac{\partial V}{\partial y} = \frac{\partial f}{\partial y}, \frac{\partial V}{\partial z} = \frac{\partial f}{\partial z}, \quad \text{X} \quad \frac{2(\pi)}{6(u)}$$

$$x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y} + z \frac{\partial V}{\partial z} = \frac{1}{6} (\pi/4)$$

$$\frac{\pi}{12}$$

1. (e) If the plane  $2x - y + cz = 0$  cuts the cone  $yz + zx + xy = 0$  in perpendicular lines, find the value of c. [10]

let the equation of line by  $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ ;

having direction ratios as  $l, m, n$ .

then this should satisfy both equation of plane and cut; as it is the intersection of two

$$2l - m + cn = 0 \quad \text{--- ①} \quad lm + mn + ln = 0 \quad \text{--- ②}$$

$$l = \frac{(m - cn)}{2} \quad \text{--- ④}$$

$$m = 2l + cn \quad \text{--- ③}$$

$$(l+n)(2l+cn) + ln = 0 \quad \{ \text{from 1, 2, 3} \}$$

$$2l^2 + (3+c)ln + cn^2 = 0 \quad \therefore \frac{l_1 l_2}{n_1 n_2} = \frac{c}{2} \quad \text{--- ⑤}$$

from 1, 2 & 4; we have

$$(m+n) \frac{(m-cn)}{2} + mn = 0$$

$$m^2 + mn(3-c) - cn^2 = 0$$

$$\frac{m_1 m_2}{n_1 n_2} = \frac{-c}{1} \quad \text{--- (6)}$$

Also, since it cuts into 2 lines

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

$$\text{or } 1 - c + \frac{c}{2} = 0$$

$$\Rightarrow c = 2$$

2. (a) Let  $F$  be a field and let  $n$  be a positive integer ( $n \geq 2$ ). Let  $V$  be the vector space of all  $n \times n$  matrices over  $F$ . Which of the following sets of matrices  $A$  and  $V$  are subspaces of  $V$ ?
- (i) All invertible  $A$ ;
  - (ii) All non-invertible  $A$ ;
  - (iii) All  $A$  such that  $AB = BA$ , where  $B$  is some fixed matrix in  $V$ ;
  - (iv) All  $A$  such that  $A^2 = A$ .

[18]

3. (a) (i) Show that the vectors

$$\alpha_1 = (1, 1, 0, 0), \alpha_2 = (0, 0, 1, 1)$$

$$\alpha_3 = (1, 0, 0, 4), \alpha_4 = (0, 0, 0, 2)$$

form a basis for  $\mathbb{R}^4$ . Find the coordinates of each of the standard basis vectors in the ordered basis  $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ .

$$(ii) \text{ Let } A = \begin{bmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{bmatrix}$$

\* Is A similar over the field R to a diagonal matrix? Is A similar over the field C to a diagonal matrix? [20]

(i) If  $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$  forms a basis, then

$$\left[ \begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 2 \end{array} \right] ; \text{ Should have rank } = 4$$

$$\begin{aligned} R_3 \rightarrow R_3 - R_1 & \sim \left[ \begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 2 \end{array} \right] \xrightarrow{R_2 \leftrightarrow R_3} \left[ \begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{array} \right] \end{aligned}$$

which is clearly have non zero row 4  
 i thus rank = 4, hence is basis

$$(a, b, c, d) = \begin{pmatrix} 1, 0, 0, 0 \end{pmatrix} = a \begin{pmatrix} 1, 1, 0, 0 \end{pmatrix} + b \begin{pmatrix} 0, 0, 1, 1 \end{pmatrix} \\ + c \begin{pmatrix} 1, 0, 0, 4 \end{pmatrix} + d \begin{pmatrix} 0, 0, 0, 2 \end{pmatrix}$$

$$a = 0, b = 0, c = 1, d = -2 \quad (0, 0, 1, -2)$$

likewise  $(0, 1, 0, 0) = a_2 \begin{pmatrix} 1, 1, 0, 0 \end{pmatrix} + b_2 \begin{pmatrix} 0, 0, 1, 1 \end{pmatrix} + c_2 \begin{pmatrix} 1, 0, 0, 4 \end{pmatrix} + d_2 \begin{pmatrix} 0, 0, 0, 2 \end{pmatrix}$

$$(0, 1, 0, 0) = \text{clearly } a_2 = 1, b_2 = 0, c_2 = -1, d_2 = 2 \quad (1, 0, -1, 2)$$

$$(0, 0, 1, 0) = a_3 \begin{pmatrix} 1, 1, 0, 0 \end{pmatrix} + b_3 \begin{pmatrix} 0, 0, 1, 1 \end{pmatrix} + c_3 \begin{pmatrix} 0, 0, 0, 4 \end{pmatrix}$$

$$(0, 0, 1, 0) = a_3 \begin{pmatrix} 1, 1, 0, 0 \end{pmatrix} + b_3 \begin{pmatrix} 0, 0, 1, 1 \end{pmatrix} + d_3 \begin{pmatrix} 0, 0, 0, 2 \end{pmatrix} \quad (0, 1, 0, -4)$$

$$a_3 = c_3 = 0, b_3 = 1, d_3 = -1/2 \quad (0, 0, 0, 1/2)$$

$$(0, 0, 0, 1) = \frac{1}{2} (0, 0, 0, 2) = \frac{1}{2} \alpha_4 \quad (0, 0, 0, 1/2)$$

$$\therefore \hat{e}_1 = \alpha_3 - 2\alpha_4 = (1, 0, 0, 0)$$

$$\hat{e}_2 = \alpha_1 - \alpha_3 + 2\alpha_4 = (0, 1, 0, 0)$$

$$\hat{e}_3 = 0\alpha_1 + \alpha_2 - \frac{\alpha_4}{2} = (0, 0, 1, 0)$$

$$\hat{e}_4 = \frac{\alpha_4}{2} = (0, 0, 0, 1)$$

(ii)  $A = \begin{bmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{bmatrix}; |A - \lambda I| = \begin{bmatrix} 6-\lambda & -3 & -2 \\ 4 & -1-\lambda & -2 \\ 10 & -5 & -3-\lambda \end{bmatrix}$

$$= (6-\lambda)[(-1-\lambda)(-3-\lambda) - 10] + 3(-12 - 4\lambda + 20) \\ - 2(-20 + 10 + 10\lambda)$$

$$(6-\lambda)(\lambda^2 + 4\lambda + 3) - (6-\lambda)10 + 24 - 12\lambda \\ + 20 - 20\lambda = 0$$

$$6\lambda^2 + 24\lambda + 18 - \lambda^3 - 4\lambda^2 - 3\lambda - 60 + 10\lambda + 44 - 32\lambda = 0 \\ \cancel{\lambda^3 - 2\lambda^2 + \lambda - 2} = 0$$

$$\lambda = 2, i, -i$$

since eigen values are both real and imaginary,

for  $\lambda = 2$ :

$$\begin{bmatrix} 4 & -3 & -2 \\ 4 & -3 & -2 \\ 10 & -5 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 ; \begin{array}{l} 4x_1 - 3x_2 - 2x_3 = 0 \\ 2x_1 - x_2 - x_3 = 0 \end{array}$$

$$\boxed{x_2 = 0} \\ \boxed{2x_1 = x_3}$$

$$\text{or } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

for  $\lambda = i$ :

$$\begin{bmatrix} 6-i & -3 & -2 \\ 4 & -1-i & -2 \\ 10 & -5 & -3-i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \begin{array}{l} (6-i)x_1 - 3x_2 - 2x_3 = 0 \\ 4x_1 - (1+i)x_2 - 2x_3 = 0 \\ 10x_1 - 5x_2 - (3+i)x_3 = 0 \end{array}$$

$$+ (2+i)x_1 + (i-2)x_2 = 0$$

$$\text{or } \boxed{x_1 = x_2}$$

$$\therefore x_2 = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} 5x_1 \\ (3+i)x_1 \\ 1 \end{bmatrix} ; \quad \boxed{\frac{5x_1}{(3+i)} = x_3}$$

consider  
in G

clearly geometric multiplicity is ~~not~~ ≠ algebraic multiplicity over  $\mathbb{C}$  (3)

Also, when eigen values all imaginary, it cannot be similar to any 'D' diagonal over  $\mathbb{R}$

3. (b) By using Lagrange Multipliers method find the maximum and minimum values of  $f(x, y, z) = xyz$  subject to the constraint  $x + 9y^2 + z^2 = 4$ . Assume that  $x \geq 0$  for this problem. Why is this assumption needed? [15]

$$f = xyz \quad ; g: x + 9y^2 + z^2 = 4 \quad \text{--- (2)}$$

$$\frac{df}{f} = \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

$$dg = dx + 18y dy + 2z dz = 0$$

∴ By lagrange's multiplier

$$df + \lambda dg = 0 \Rightarrow$$

$$\frac{dx}{x} + \lambda dx = 0; \quad \frac{dy}{y} + \lambda(18y) dy = 0$$

$$\frac{dz}{z} + \lambda 2z dz = 0$$

$$\lambda = -\frac{1}{x} \quad \text{or} \quad x = -1/\lambda$$

$$x + 9y^2 + z^2 = 4 \Rightarrow$$

$$\lambda - \frac{1}{x} = 4$$

$$y^2 = \pm \frac{1}{18\lambda}, \quad z^2 = \frac{\pm 1}{2\lambda}$$

$$\begin{aligned} \frac{1}{x} + \frac{9}{18}x + \frac{1}{2\lambda} &= 4 \\ y^2 &= \frac{1}{18\lambda}, \quad z^2 = \frac{1}{2\lambda} \end{aligned}$$

$$9y^2 = \frac{-1}{2\lambda}; \quad z^2 = \frac{-1}{2\lambda}$$

$$x + 9y^2 + z^2 = -\frac{2}{\lambda} = 4 \Rightarrow \lambda = (-4)^1 \quad \text{or} \quad \lambda = -1/4$$

if  $x < 0$ , then  $\lambda$  is positive  $\rightarrow y, z$  are

which does not satisfy 2, hence  $x > 0$  is only cond.

to obtain max or minimum value of function

$$\therefore x = 2, \quad y = \pm \frac{1}{3}, \quad z = \pm 1$$

if  $x = 2, y = \frac{1}{3}, z = 1$ ; then

$$f_x = \frac{d}{dx} (xyz) = yz + xy \frac{dz}{dx}$$

$$\text{from 2; } 1 + \frac{d}{dx}(2z) = 0 \quad \text{or} \quad \frac{dz}{dx} = -\frac{1}{2z}$$

$$f_x = yz - \frac{xy}{2z}$$

$$f_{xx} = \frac{y}{dx} - \frac{y}{2z} + \frac{xy}{2z^2} \left( \frac{dz}{dx} \right)$$

$$\begin{aligned} f_{xx} &= -\frac{y}{2z} - \frac{y}{2z} + \left( -\frac{xy}{4z^3} \right) \\ &= -\frac{y}{z} \left( 1 + \frac{x}{4z^2} \right) \end{aligned}$$

$$\text{if } x = 2, y = \frac{1}{3}, z = 1; \text{ then}$$

$$\{ f_{xx} \leq 0; \text{ implying } \text{here it is max} \}$$

$$\text{for minimum if } x = 4; \text{ then } 9y^2 + z^2 \geq 0$$

$$f_{min} = xyz = 0$$

$$f_{min} = -\frac{1}{3}$$

$$(2, \frac{1}{3}, 1) \quad \text{and} \quad (2, -\frac{1}{3}, 1)$$

even for this

$$f_{max} = \frac{2}{3}$$

$f_{max}$  is achieved

3. (c) show that the locus of points from which three mutually perpendicular tangents can be drawn to the paraboloid  $ax^2 + by^2 = 2z$  is given by  
 $ab(x^2 + y^2) - 2(a+b)z - 1 = 0$  [15]

Locus of points from which 3 mutually  
 to tangents can be drawn is equation of  
 enveloping cone [given by  $s_1 = T$  further as 3 tr generators]  
 let the point be  $(x_1, y_1, z_1)$

$$\text{then } s_1 = ax_1^2 + by_1^2 - 2z_1$$

$$S: ax^2 + by^2 + cz^2 - 2z$$

$$T: ax_1x + by_1y - (z + z_1)$$

$$(ax_1^2 + by_1^2 - 2z_1)(ax^2 + by^2 - 2z) = (ax_1x + by_1y - (z + z_1))^2 \quad (1)$$

$$\text{coeff of } x^2 = a^2x_1^2 + aby_1^2 - 2az_1 - a^2x^2$$

$$\text{coeff of } y^2 = b^2y_1^2 + abx_1^2 - 2bz_1 - b^2y^2$$

$$\text{coeff of } z^2 = -1$$

for this (1) to have 3 tr generators, coefficient of  
 $x^2 + \text{coeff of } y^2 + \text{coeff of } z^2 = 0$

$$aby_1^2 + abx_1^2 - 2az_1 - 2bz_1 - 1 = 0$$

$$\text{or } ab(x_1^2 + y_1^2) - 2(a+b)z_1 - 1 = 0 \therefore$$

is the required locus



4. (a) Let  $T$  be the linear transformation from  $\mathbb{R}^3$  into  $\mathbb{R}^2$  defined by  $T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1)$ .

(i) If  $\beta$  is the standard ordered basis for  $\mathbb{R}^3$  and  $\beta'$  is the standard ordered basis for  $\mathbb{R}^2$ , what is the matrix of  $T$  relative to the pair  $\beta, \beta'$ ?

(ii) If  $\beta = \{\alpha_1, \alpha_2, \alpha_3\}$  and  $\beta' = \{\beta_1, \beta_2\}$ , where

$\alpha_1 = (1, 0, -1), \alpha_2 = (1, 1, 1), \alpha_3 = (1, 0, 0), \beta_1 = (0, 1), \beta_2 = (1, 0)$  what is the matrix of  $T$  relative to the pair  $\beta, \beta'$ ? [16]

$$(a) (i) \quad \beta = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\};$$

$$\beta' = \{(1, 0), (0, 1)\}.$$

$$T: [\beta_1, \beta_2] \ni T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1) \\ T(1, 0, 0) = (1, 0) = 1 \cdot (1, 0) + (-1) \cdot (0, 1)$$

$$T(0, 1, 0) = (1, 0) = 1 \cdot (1, 0) + 0 \cdot (0, 1)$$

$$T(0, 0, 1) = (0, 2) = 0 \cdot (1, 0) + 2 \cdot (0, 1)$$

$$T: [\beta, \beta'] := \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \quad T$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix} \text{ is the reqd matrix } T$$

$$(ii) \quad \text{if } x_1 = (1, 0, -1), x_2 = (1, 1, 1), \\ x_3 = (1, 0, 0)$$

$$\beta_1 = (0, 1), \beta_2 = (1, 0)$$

$$T(x_1) = T(1, 0, -1) = (1, -3) \\ -3(0, 1) + 1(1, 0)$$

$$\tau(\alpha_2) = (1+1, 2-1) = (2, 1) = 1(0, 1) + 2(1, 0)$$

$$\tau(\alpha_3) = \tau(1, 0, 0) = (1, -1) = -1 \cdot (0, 1) + 1 \cdot (1, 0)$$

$$\tau[B; B'] = \begin{bmatrix} -3 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

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4. (b) (i) Show that  $\frac{2}{\pi} < \frac{\sin x}{x} < 1, 0 < x < \pi/2$ .

(ii) Determine  $\lim_{x \rightarrow \left(\frac{\pi}{2}-0\right)} \left(\frac{\pi}{2}-x\right)^{\tan x}$ .

(iii) Find the volume bounded by the cylinder  $x^2 + y^2 = 4$  and the planes  $y + z = 4$  and  $z = 0$ . [5+5+8=18]

$$(b)(i) f(x) = \frac{\sin x}{x} ; \text{ where } 0 < x < \pi/2$$

$$f'(x) = \frac{x \cos x - \sin x}{x^2}$$

for  $f'(x) = 0 \rightarrow \boxed{\tan x = x}$

this is satisfied at  $x=0$ .

$$\left. \begin{array}{l} f''(x) \leq 0 \\ f''(x) = \frac{2x^2 \{ \cos x - \tan x - x \sin x \} - 2x(x \cos x - \sin x)}{(x^2)^2} \end{array} \right\}$$

$f(x)$  is decreasing function and  $x=0$  is min at  $x=0$ .

$$- f(0) = \lim_{x \rightarrow 0} \frac{\sin(0)}{0} = 1$$

$$f(\pi/2) = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin x}{x} = \frac{2}{\pi}$$

$$\frac{2}{\pi} < \frac{\sin x}{x} < 1,$$

$$(ii) \lim_{x \rightarrow \frac{\pi}{2}^-} \left(\frac{\pi}{2} - x\right)^{\tan x} \text{ as } x \rightarrow \left(\frac{\pi}{2} - 0^-\right)$$

$y = \left(\frac{\pi}{2} - x\right)^{\tan x}$ ; it is of form:  $0^\infty$  as  $x \rightarrow \frac{\pi}{2}^-$

$$\log y \left[ \log \left( \frac{\pi}{2} - x \right) \right] \tan x$$

$$= \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{\log \left( \frac{\pi}{2} - x \right)}{\tan x} \quad \left\{ \begin{array}{l} \frac{\infty}{\infty} \\ \text{form} \end{array} \right.$$

let  $x = \frac{\pi}{2} - h$ , where  $h \rightarrow 0^+$

$$= \lim_{h \rightarrow 0^+} \frac{\log h}{\tan h} = \lim_{h \rightarrow 0^+} \frac{\log h}{\sinh}$$

$$= \lim_{h \rightarrow 0^+} \left( h - \frac{h^2}{2} + \frac{h^3}{3} - \dots \right) \cos \theta \quad \equiv 1 \quad X$$

$$h - \frac{h^3}{3!} + \frac{h^5}{5!} - \dots$$

??

b(iii) volume bounded =  $\int dz dy dx$

$$= \iint_{x=0}^{4-y} dz dy dx$$

wide:  $x^2 y^3 z^4$

$$V = \iint (4-y) dz dy$$

using polar co-ordinates; we have  
 $x = r \cos \theta, y = r \sin \theta$

$$V = \iint_0^{2\pi} (4 - r \sin \theta) r dr d\theta$$

$$= \iint_0^{2\pi} 4r dr d\theta = 16\pi \text{ m } \underline{\text{volume}}$$

4. (c) Prove that the projections of the generators of a hyperboloid on coordinate plane are tangents to the section of the hyperboloid by that plane. [16]

let the hyperboloid be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1; \text{ any generator to this is}$$

given by  $\frac{x = a \cos \theta}{+ a \sin \theta} = \frac{y = b \sin \theta}{- b \cos \theta} = \frac{z = c}{+ c}$ .

at any point  $(a \cos \theta, b \sin \theta, c)$  of the

hyperboloid

for  $z=0$ , co-ordinates; we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x = a \cos \theta}{- a \sin \theta} = \frac{y = b \sin \theta}{- b \cos \theta}$$

$$\frac{x}{a \sin \theta} + \frac{y}{b \cos \theta} = \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta}$$

$$\frac{x \cos \theta}{a \cos \theta} + \frac{y \sin \theta}{b \sin \theta} = 1$$

and any tangent at  $(a \cos \theta, b \sin \theta)$  is

②

given by  $\frac{x \cdot x_1}{a^2} + \frac{y \cdot y_1}{b^2} = 1$  : for ellip sec;

$$\therefore \frac{x}{a^2} \cos \theta + \frac{y}{b^2} \cdot b \sin \theta = 1 \Rightarrow \frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1$$

$$\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1$$

here for  $z=0$ , it forms the tangent plane for

$$\text{Ellipse: } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

Likewise for  $y=0$ ;  $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$  ;  
any point is given by  $[a \sec \theta, c \tan \theta]$

Tangent is given by

$$\frac{x \sec \theta}{a^2} - \frac{z \cdot c \tan \theta}{c^2} = 1$$

or from ①

$$\Rightarrow \frac{x \sec \theta}{a} - \frac{z \tan \theta}{c} = 1$$

$$\frac{x \sec \theta}{a} = \frac{z}{c} \Rightarrow \frac{x}{a \sin \theta} - \frac{z}{c} = \frac{\cos \theta}{\sin \theta}$$

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$$\frac{x \sec \theta}{a} - \frac{z \tan \theta}{c} = 1 \quad \text{hence ④ & ⑤ are same}$$

$$\frac{x \sec \theta}{a} - \frac{z \tan \theta}{c} = 1$$

Likewise for  $x=0$ ;  $\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ ; any point is  
 $(b \sec \theta, 0, z)$ ; & tangent  $y \sec \theta - \frac{z \tan \theta}{c} = 1$  & from ①

it can be shown again

## SECTION - B

5. (a) (i) Solve  $(2\sqrt{xy} - x)dy + y dx = 0$

(ii) Solve  $(y + y^3/3 + x^2/2) dx + (1/4) \times (x + xy^2) dy = 0.$

[10]

$$(i) N = 2\sqrt{xy} - x; M = y$$

$$\frac{\partial M}{\partial y} = 1; \quad \frac{\partial N}{\partial x} = \frac{y}{x} - 1$$

$$\text{or } 2\sqrt{xy} dy + y d(x/y) = 0$$

$$2\sqrt{y} dy + \left(\frac{\sqrt{y}}{\sqrt{x}}\right)(y^{3/2}) d(x/y) = 0$$

$$\frac{2}{y} dy + \sqrt{\frac{y}{x}} d(x/y) = 0$$

$$2\log y + 2\log \sqrt{xy} = 0 + \text{constant}$$

$$\text{or } \boxed{\log y + \sqrt{xy} = C}$$

$$(ii) M = y + y^3/3 + x^2/2; \quad N = \frac{(x + xy^2)}{4}$$

$$\frac{\partial M}{\partial y} = 1+y^2; \quad \frac{\partial N}{\partial x} = \frac{1+y^2}{4}$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{3(1+y^2)}{4} \cdot \frac{1}{x(1+y^2)/4} = \frac{3}{x} \quad (\text{I.V.D})$$

$$\text{Integrating factor} = \int e^{3/x} dx$$

$$= x^3$$

Multiply entire eqn by  $x^3$

$$(x^3y + \frac{x^3y^3}{3} + \frac{x^5}{2})dx + \frac{(x^4 + x^4y^2)}{4}dy = 0$$

$$\frac{d(x^4y^3)}{12} + \frac{d(x^4y)}{4} + \frac{x^5dx}{2} = 0$$

or  $\left[ \frac{x^4y^3}{12} + \frac{x^4y}{4} + \frac{x^6}{12} \right] = \text{constant}$

09

5. (b) (i) Prove  $L\left\{\frac{\cos at - \cos bt}{t}\right\} = \frac{1}{2} \log \frac{s^2 + b^2}{s^2 + a^2}$

(ii) Evaluate  $\int_0^\infty t^3 e^{-t} \sin t dt.$

[10]

(i) using  $L\left\{\frac{f(t)}{t}\right\} = \int_p^\infty f(\beta) d\beta$

where  $f(\beta) = L^*\{f(t)\}$

$$\therefore L\{\cos at - \cos bt\} = \frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2} = f(s)$$

$$\begin{aligned} \int_0^\infty t^3 e^{-t} \sin t dt &= \int_p^\infty \int_0^\infty \frac{s}{s^2 + a^2} ds - \int_p^\infty \frac{s}{s^2 + b^2} ds \\ &= \frac{1}{2} \log \left( \frac{s^2 + b^2}{s^2 + a^2} \right) \end{aligned}$$

$$(ii) \int_0^{\infty} t^3 e^{-st} \sin t \cdot dt = f(s) \quad \text{--- } \textcircled{1}$$

$$\text{let } f(s) = \int_0^{\infty} e^{-st} \cdot t^3 \sin t \, dt$$

$$L\{t^3 \sin t\} = \frac{(-1)^3}{ds^3} \left( L\{\sin t\} \right)$$

$$L\{\sin t\} = \frac{1}{s^2 + 1}$$

$$L\{t^3 \sin t\} = \frac{(-1)^3}{ds^3} \left( \frac{1}{s^2 + 1} \right)$$

$$\frac{d}{ds} \left( \frac{1}{s^2 + 1} \right) = \frac{-2s}{(s^2 + 1)^2}$$

$$\frac{d}{ds^2} \left( \frac{1}{s^2 + 1} \right) = -\frac{d}{ds} \left( \frac{2s}{(s^2 + 1)^2} \right) = \frac{-2}{(s^2 + 1)^2} + \frac{(4s)(2s)}{(s^2 + 1)^3}$$

$$\frac{d}{ds^3} \left( \frac{1}{s^2 + 1} \right) = \frac{(-2)(-2)(2s)}{(s^2 + 1)^3} + \frac{16s}{(s^2 + 1)^3} - \frac{(24)(2s^3)}{(s^2 + 1)^4}$$

$$\text{or } L\{t^3 \sin t\} = - \left\{ \frac{8s}{(s^2 + 1)^3} + \frac{16s}{(s^2 + 1)^3} - \frac{48s^3}{(s^2 + 1)^4} \right\}$$

put  $s=1$  ;

$$\text{from l.i. } \int_0^{\infty} t^3 e^{-st} \sin t \, dt = f(s) \\ = - \left\{ \frac{2^4}{2^3} - \frac{48}{2^4} \right\} = \underline{\underline{0}}$$

5. (c) A heavy uniform rod rests with one end against a smooth vertical wall and with a point in its length resting on a smooth peg; find the position of equilibrium and show that it is unstable. [10]

Let AB be the rod; of length  $(2l)$  of Mass  $M$ ; and OP be peg of length  $(c')$

the force acts on  $G_1$ ; the distance between the fixed points is  $60$

$$\bar{z} = OG_1 = (AG) - AO$$

$$\bar{z} = l \cos \theta - \cosec \theta$$

$$\frac{dz}{d\theta} = -l \sin \theta + c \cot \theta \cdot \cosec \theta$$

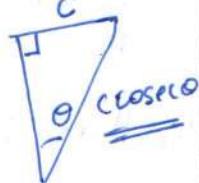
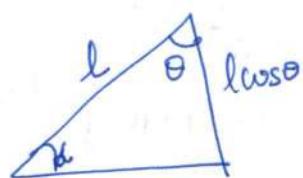
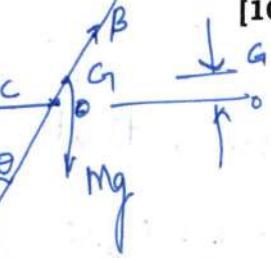
$$\frac{dz}{d\theta} = 0 \text{ gives position of equilibrium}$$

$$\text{or } l \sin^3 \theta = c \cos \theta$$

$$\frac{d^2 z}{d\theta^2} = -l \cos \theta + c \frac{d}{d\theta} \left( \frac{\cos \theta}{\sin^2 \theta} \right)$$

$$= -l \cos \theta - 2 \cos^2 \theta - \frac{c \sin \theta}{\sin^3 \theta}$$

clearly  $\frac{d^2 z}{d\theta^2} < 0$ ; hence its unstable equilibrium



OF -

5. (d) A particle moves with a central acceleration which varies inversely as the cube of the distance. If it be projected from an apse at a distance  $a$  from the origin with a velocity which is  $\sqrt{2}$  times the velocity for a circle of radius  $a$ , show that the equation to its path is  $r \cos(\theta/\sqrt{2}) = a$ . [10]

Given that  $P(\text{accelerated}) \propto \frac{1}{r^3}$   
 or  $P = M/r^3$ ,  $P = Mu^3$

$$\frac{P}{h^2 u^2} = u + \frac{du}{d\theta} \Rightarrow \frac{Mu}{h^2} = u + \frac{du}{d\theta}$$

multiplying by  $(\frac{2du}{d\theta})$  and integrating  
 $\frac{Mu^2}{h^2} = u^2 + (\frac{du}{d\theta})^2 \Rightarrow A + Mu^2 = h^2(u^2 + (\frac{du}{d\theta})^2) = V^2$  ①

further; velocity at  $u=1/a$  is

$\sqrt{2} \times (\text{velocity for circle with radius } a)$   
 velocity of circle radius  $a$  ( $V$ ) :  $\frac{V^2}{a} = P|_{r=a}$

$$V^2 = \frac{aM}{a^3} \Rightarrow V = \frac{M}{a^2} \quad (V = \sqrt{2} M/a)$$

from ①:  $A + \frac{M}{a^2} = \frac{h^2}{a^2} = \frac{2M}{a^2}$ ; as  $\frac{du}{d\theta} = 0$   
 at apse

$$A = M/a^2 ; \quad h = \sqrt{2} M$$

$$M(u^2 + \frac{1}{a^2}) = 2M(u^2 + (\frac{du}{d\theta})^2) \Rightarrow \left(\frac{1}{a^2} - u^2\right) = \frac{2(du)^2}{d\theta}$$

$$\sqrt{\frac{1}{a^2} - u^2} du = \frac{d\theta}{\sqrt{2}} \Rightarrow \frac{(\cos^2 u a)}{a} = \frac{\theta}{\sqrt{2}} + B$$

at  $u = \frac{1}{a}; B = 0; B = 0$ ;  $\cos^2 a/r = \theta/\sqrt{2}$   
 $\Rightarrow r \cos(\theta/\sqrt{2}) = a$

5. (e) Verify Green's theorem in the plane for

$$\int_C (x^2 - xy^3) dx + (y^2 - 2xy) dy,$$

where C is the square with vertices (0, 0), (2, 0), (2, 2), (0, 2). [10]

$$\textcircled{1} \quad \int_C M dx + N dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

$$\text{here } N = y^2 - 2xy, \quad M = x^2 - xy^3$$

$\therefore$

$$\textcircled{2} \quad \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \int_0^2 \int_0^2 (-2y + 3xy^2) dx dy$$

$$= \int_0^2 \left[ \left( -y^2 + xy^3 \right) \Big|_0^2 \right] dx = \int_0^2 (-4 + 8x) dx = (-4x + 4x^2) \Big|_0^2$$

$$= \cancel{-24} \quad \cancel{8}$$

calculating line integral,

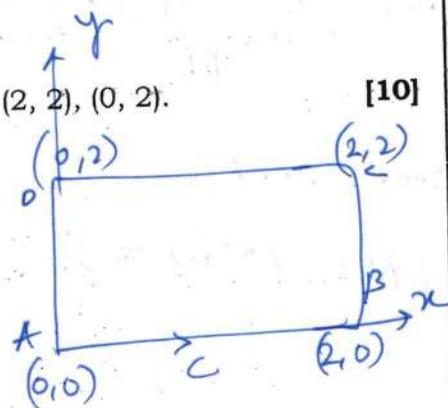
$$\int_C M dx + N dy = \int_0^2 x^2 dx = \frac{8}{3} \quad \begin{cases} \text{Along AB:} \\ dy = 0 \\ y = 0 \end{cases}$$

$$\int_{AB} M dx + N dy = \int_0^2 (y^2 - 4y) dy \quad \textcircled{2}$$

$$= \frac{8}{3} - 8 = \cancel{\frac{8}{3} - 8}$$

$$\int_{CD} M dx + N dy = \int_0^2 (x^2 - 8x) dx$$

$$= - \int_0^2 (x^2 - 8x) dx = - \left\{ \frac{8}{3} - 16 \right\} \quad \textcircled{3}$$



Along DA:  $\int_M dx + N dy = - \int y^2 dy = -8/3$   $\left\{ \begin{array}{l} x=0 \\ dx=0 \end{array} \right.$

From 1, 2, 3, 4

~~$$\int_M dx + N dy = \frac{8}{3} + \frac{8}{3} - 8 + 16 - \frac{8}{3} = \frac{8}{3}$$~~

hence  $\int_M dx + N dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$ ;  
 & verified Green's theorem in plane

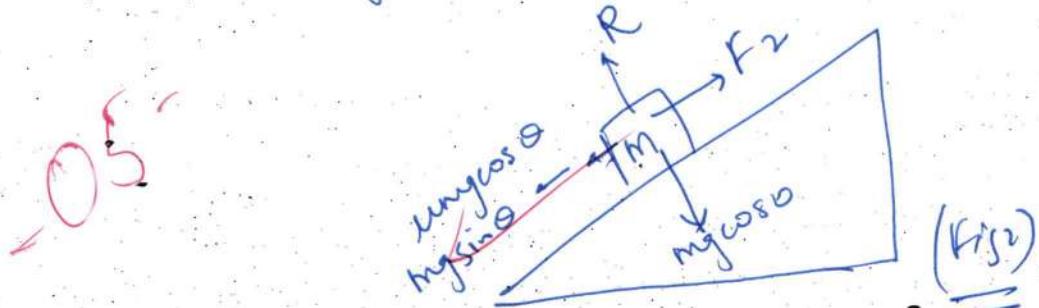
6. (a) (i) Find the equation of the family of oblique trajectories which cut the family of concentric circles at  $30^\circ$ .  
 (ii) Reduce the equation  $x^2 p^2 + py(2x+y) + y^2 = 0$  where  $p = dy/dx$  to Clairaut's form and find its complete primitive and its singular solution. [7+10=17]

7. (a) A weight of 60 kg is on the point of motion down a rough inclined plane when supported by a force of 24 kg wt acting parallel to the plane along a line of greatest slope, and is on the point of motion up the plane when pulled in the same direction by force of 36 kg wt. Find the co-efficient of friction and the inclination of the plane. [17]

$$F_1 + mg \sin \theta = \mu mg \cos \theta \quad (1)$$

$$F_2 = mg \sin \theta + \mu mg \cos \theta \quad (2)$$

(Fig 1)



where  $F_1$  is force acting down the plane

$$F_1 = 24g;$$

$F_2$  is force acting up the plane:  $F_2 = 36g$

Subtracting 2 from ①

??

$$F_1 - F_2 + 2mg \sin \theta = 0$$

$$\text{or } 2mg \sin \theta = 12g$$

$$m = 60 \text{ kg}$$

$$2g \sin \theta = 11/10$$

① ② ③

Adding

$$F_1 + F_2 = 2mg \cos \theta$$

$$60g = 2\mu (60)g \cos \theta$$

$$\mu \cos \theta = 1/2$$

$$\sin \theta = \frac{1}{10}, \cos \theta = \frac{\sqrt{99}}{10} = 0.994$$

$$\mu = \frac{1}{\sqrt{99}} = 0.11547$$

$$\text{Angle of inclination} = \sin^{-1}(1/10) = 5.739^\circ$$

$$\sin^{-1}(1/2) \Rightarrow \alpha = 30^\circ$$

7. (b) A heavy particle is attached to one end of an elastic string, the other end of which is fixed. The modulus of elasticity of the string is equal to the weight of the particle. The string is drawn vertically down till it is four times its natural length and then let go. Show that the particle will return to this point in time  $\sqrt{\left(\frac{a}{\lambda}\right)\left[\frac{4\pi}{3} + 2\sqrt{3}\right]}$ , where  $a$  is the natural length of the string.

[17]

Given that  $\lambda = \text{modulus of elasticity}$

$w$  is weight of particle

$$\therefore w = \frac{\lambda s}{a} (x - a)$$

$$\text{or extenstion} = \frac{\lambda s}{a} = AB$$

$$\sqrt{\left(\frac{a}{\lambda}\right)\left[\frac{4\pi}{3} + 2\sqrt{3}\right]} = \sqrt{(2w)} = 2mg$$

Tension in the string is balanced

by weight and spring force

$$\therefore m\frac{d^2x}{dt^2} = mg + \lambda \left(\frac{x - a}{a}\right) \quad \left\{ \text{as } \frac{\lambda}{2} = mg \right\}$$

$$\Rightarrow \frac{d^2x}{dt^2} = -\frac{2mgx}{a}$$

$$\frac{d^2x}{dt^2} = -\frac{2g}{a}x$$

is governing law

$$\frac{dx}{dt} = -\frac{2g}{a}(x^2) + A$$

(i) Then the velocity is 0 when at

$$AB = a/2 \text{ and } OA = 0; BC = 4a \Rightarrow$$

$$\frac{dx}{dt} = +\frac{2g}{a} \left( \left(\frac{5a}{2}\right)^2 - x^2 \right)$$

$$BC = \frac{5a}{2}$$

$$\int_{-\frac{a}{2}}^{\frac{5a}{2}} \frac{dx}{\sqrt{\left(\frac{5a}{2}\right)^2 - x^2}} = \int \sqrt{\frac{2g}{l}} dt \quad \text{--- (2)}$$

$$\sin^{-1}\left(\frac{x}{\frac{5a}{2}}\right) \Big|_{-\frac{a}{2}}^{\frac{5a}{2}} = \sqrt{\frac{2g}{l}} t$$

$$\sin^{-1}(1) - \sin^{-1}(-\frac{1}{5}) = \sqrt{\frac{2g}{l}} t \quad \text{--- (3)}$$

this  $t$  is time required from  $C$  to  $t$ .

after  $A$ , there is no extension. now

$$V_A = \sqrt{\frac{2g}{a}} \sqrt{\left(\frac{5a}{2}\right)^2 - \left(\frac{a}{2}\right)^2} = \sqrt{12ga} \quad ; = 2\sqrt{3}ag$$

$$t_2 = \text{time after } A \text{ to return to } A \Rightarrow V - u = gt$$

$$\text{or } t_2 = 2\sqrt{3} \frac{a}{g}$$

$$\text{Total time} = 2(t_1 + t_2) \quad (t_1 \text{ given from (3)})$$

7. (c) A particle is projected with a velocity  $u$  from a point on an inclined plane whose inclination to the horizontal is  $\beta$ , and strikes it at right angles. Show that

$$(i) \text{ The time of flight is } \frac{2u}{g\sqrt{1+3\sin^2\beta}}$$

$$(ii) \text{ The range on the inclined plane is } \frac{2u^2}{g} \cdot \frac{\sin\beta}{1+3\sin^2\beta}$$

and (iii) The vertical height of the point struck, above the point of projection is

$$\frac{2u^2 \sin^2\beta}{g(1+3\sin^2\beta)}$$

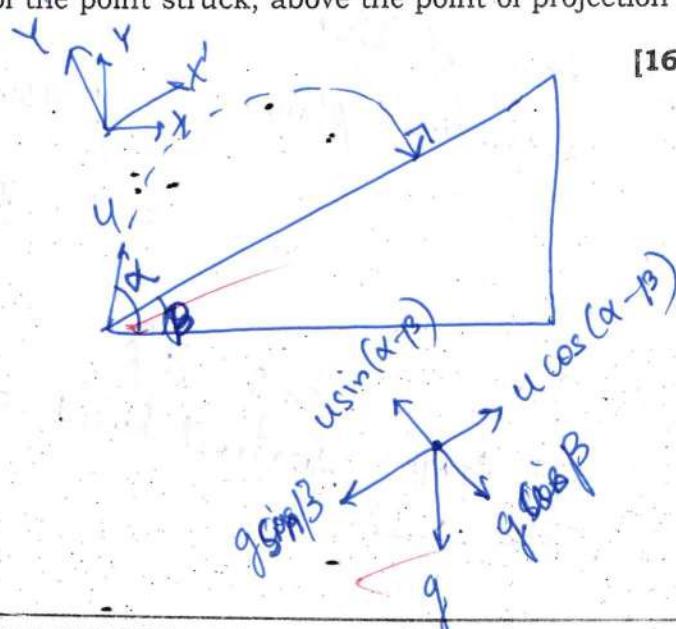
(Q) (i)

time of flight

$$S = ut - \frac{1}{2}gt^2$$

in  $y'$ ,  $x'$  axis system

$$t = \frac{2u \sin(\alpha - \beta)}{g \cos \beta} \quad \text{and}$$



$$\underline{V_y = 0} ; \therefore V_x - U_x = g_x t$$

$$\text{or } t = \frac{U \cos(\alpha - \beta)}{g \sin \beta}$$

this both time should be equal:  $\frac{U \cos(\alpha - \beta)}{g \sin \beta} = \frac{2U \sin(\alpha - \beta)}{g \cos \beta}$

$$\therefore \cot 2 \tan(\alpha - \beta) = \cot \beta$$

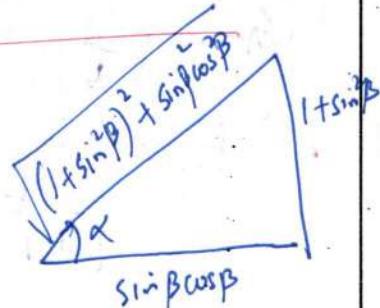
$$\frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} = \cot \beta - 2 \tan \alpha^2 - \tan \beta = \cot \beta + \tan \alpha$$

$$2 \tan \alpha = \left( \frac{1 + 3 \sin^2 \beta}{\sin \beta \cos \beta} \right)$$

$$(i) \text{ time of flight} = \frac{U (\cos \alpha \cos \beta + \sin \alpha \sin \beta)}{g \sin \beta} = U (\cos \alpha \cdot \cot \beta + \sin \alpha)$$

$$t = \frac{U}{g \sin \beta} \left( \frac{\cos \beta \cdot \sin \beta \cos \beta}{\sqrt{1 + 3 \sin^2 \beta}} + (1 + 3 \sin^2 \beta) \sin \beta \right)$$

$$\cos \alpha = \frac{\sin \beta \cos \beta}{\sqrt{1 + 3 \sin^2 \beta}}, \sin \alpha = \frac{1 + 3 \sin^2 \beta}{\sqrt{1 + 3 \sin^2 \beta}}$$



$$t = \frac{2U}{g \sqrt{1 + 3 \sin^2 \beta}}$$

$$(ii) \text{ Range of flight on inclined plane: } = U \cos(\alpha - \beta) t - \frac{g \sin \beta \cdot t^2}{2}$$

$$= \frac{1}{2} g \sin \beta \cdot t^2$$

$$= \frac{1}{2} g \sin \beta \frac{4U^2}{g^2 (1 + 3 \sin^2 \beta)} = \frac{2U^2 \sin \beta}{g (1 + 3 \sin^2 \beta)}$$

(iii)



$$\sin \beta = H/R$$

$$\text{vertical height} = R \sin \beta$$

$$= \frac{2U^2 \sin^2 \beta}{g (1 + 3 \sin^2 \beta)}$$

$$2\sqrt{xy} - \alpha^2 d(y/n)$$

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