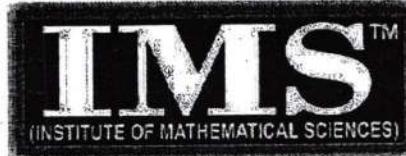


A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET



184
 (250)

MAINS TEST SERIES-2020

(OCT. TO JAN.-2020-21)

IAS/IFoS
MATHEMATICS
 Under the guidance of K. Venkanna

FULL SYLLABUS (PAPER-I)

DATE : 06-DEC-2020

Common Test
 Test-15 for Batch-I
 &
 Test-7 for Batch-II

Time: 3 Hours**Maximum Marks: 250****INSTRUCTIONS**

- This question paper-cum-answer booklet has 50 pages and has 43 PART/SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
- Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
- A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated."
- Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
- Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any **THREE** of the remaining questions selecting at least **ONE** question from each Section.
- The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
- Symbols/notations carry their usual meanings, unless otherwise indicated.
- All questions carry equal marks.
- All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
- All rough work should be done in the space provided and scored out finally.
- The candidate should respect the instructions given by the invigilator.
- The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any page from this booklet.

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name

Suryabhan

Roll No.

ORN

Test Centre

Offline

Medium

English

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the invigilator

IMPORTANT NOTE:

Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED
1	(a)			
	(b)			07
	(c)			08
	(d)			08
	(e)			
2	(a)			17
	(b)			12
	(c)			14
	(d)			
3	(a)			
	(b)			
	(c)			
	(d)			
4	(a)			16
	(b)			66
	(c)			14
	(d)			
5	(a)			08
	(b)			08
	(c)			08
	(d)			07
	(e)			05
6	(a)			
	(b)			
	(c)			
	(d)			
7	(a)			13
	(b)			15
	(c)			18
	(d)			
8	(a)			
	(b)			
	(c)			
	(d)			
Total Marks				184

1. (b) Let $T : \mathbb{C} \rightarrow M_{2,2}$ be given by

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a+b & a+b+c \\ a+b+c & a+d \end{bmatrix}. \text{ Find a basis of } R(T). \text{ Is } T \text{ surjective?}$$

[10]

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = a \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + b \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Probable basis of Range of T is $\left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

however this basis may have dependent element

for ex: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Reduced independent basis of $R(T)$ is $\left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}$

~~∴ T is not surjective~~; as
~~for $x = a \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + b \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a+b+c & a+b \\ a+b & a+d \end{pmatrix}$~~
~~there exists $a, b, c, d \in \mathbb{K}$~~

1. (c) (i) Evaluate $\left(\frac{\tan x}{x}\right)^{1/x^2}$, ($x \rightarrow 0$)

(ii) If $z = (x+y) + (x+y)\varphi(y/x)$, prove that

$$x\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y \partial x}\right) = y\left(\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x \partial y}\right) \quad [10]$$

(c)(i) $y = \left(\frac{\tan x}{x}\right)^{1/x^2} \quad (1)^\infty \text{ form as } x \rightarrow 0.$

$$\log y = \frac{1}{x^2} (\log(\tan x) - \log x) \quad (0)$$

$$= \frac{1}{x^2} \log(\tan x/x) \quad \begin{cases} \text{Applying} \\ \text{'Hospital's rule}' \end{cases}$$

$$\lim_{x \rightarrow 0} \log y = \lim_{x \rightarrow 0} \frac{1}{x^2} \frac{1}{(\tan x)/x} \times \frac{x \sec^2 x - \tan x}{2x}$$

$$= \lim_{x \rightarrow 0} \frac{x \sec^2 x - \tan x}{2x^2 \cdot \tan x} \quad (0) \text{ form}$$

$$= \lim_{x \rightarrow 0} \frac{\sec^2 x + 2\sec^2 x \cdot \tan x \cdot x - \sec^3 x}{4x \tan x + 2x^2 \sec^2 x} \quad = \lim_{x \rightarrow 0} \frac{2\sec^3 x \cdot \tan x}{4 \tan x + 2x \sec^2 x}$$

$$= \lim_{x \rightarrow 0} \frac{4\sec^2 x \tan x + 2\sec^4 x}{4\sec^2 x + 2\sec^2 x + 2x(2\sec^2 x \cdot \tan x)} = \frac{1}{3}$$

or $y = e^{Y_3}$

(ii) by cauchy euler's formula for homogeneous equation with degree $n=1$; we $\frac{x \frac{d^2 z}{dx^2} + y \frac{d^2 z}{dy^2}}{2y^2} = n z = z \quad \text{--- (1)}$

$$x \frac{d^2 z}{dx^2} + 2xy \frac{d^2 z}{dxdy} + y \frac{d^2 z}{dy^2} = n(n-1)z = 0 \quad \text{--- (2)}$$

$$x \frac{d^2 z}{dx^2} + 2xy \frac{d^2 z}{dxdy} + y \frac{d^2 z}{dy^2} = n(n-1)z = 0 \quad \text{Diff (1) w.r.t } x \text{ and } y \text{ are.}$$

$$x \frac{d^2 z}{dx^2} + 2y \frac{d^2 z}{dxdy} + y \frac{d^2 z}{dy^2} = 2y \quad \frac{d^2 z}{dxdy} + y \frac{d^2 z}{dy^2} = \frac{2y}{x} \quad \text{--- (3)}$$

$$x \frac{d^2 z}{dx^2} + 2y \frac{d^2 z}{dxdy} + y \frac{d^2 z}{dy^2} = 2y \quad \frac{d^2 z}{dxdy} + y \frac{d^2 z}{dy^2} = \frac{2y}{x} \quad \text{--- (4)}$$

$$\text{from (3) & (4) clearly we get the expression.}$$

1. (d) For the function

$$f(x,y) = \begin{cases} \frac{x^2 - x\sqrt{y}}{x^2 + y}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Examine the continuity and differentiability.

[10]

$$f(x,y) = \begin{cases} \frac{x^2 - x\sqrt{y}}{x^2 + y} ; & (x,y) \neq (0,0) \\ 0 ; & (x,y) = (0,0) \end{cases}$$

~~(x,y)~~
let us take $y = mx^2$; as $x \rightarrow 0, y \rightarrow 0$

and check for continuity;

$$\lim_{(x,y) \rightarrow (0,0)} f = \lim_{x \rightarrow 0} \frac{x^2 - x(\sqrt{mx^2})}{x^2 + mx^2} = \frac{1 - \sqrt{m}}{1+m}$$

as this value varies as m varies; that is
 function taken different values in vicinity of $(0,0)$
 hence not continuous
 A Non continuous function is non differentiable

1. (e) If the axes are rectangular, find the S.D. between the lines $y = az + b$, $z = \alpha x + \beta$ and $y = a' z + b'$, $a = \alpha' x + \beta'$?

Also deduce the condition for the lines to intersect.

[10]

$$\text{Let } L_1: \frac{x + \beta/\alpha}{(1/\alpha)} = \frac{(y - b)/a}{z}$$

$$a \left[\frac{x + \beta/\alpha}{(1/\alpha)} \right] = \frac{y - b}{z} = ?$$

$$L_2: x$$

δ

2. (a) (i) Suppose that $\{v_1, v_2, v_3, \dots, v_n\}$ is a set of vectors. Prove that $\{v_1 - v_2, v_2 - v_3, v_3 - v_4, \dots, v_n - v_1\}$ is a linearly dependent set.
- (ii) Suppose that $\{v_1, v_2, v_3, v_4\}$ is a linearly independent set in \mathbb{C}^{35} . Prove that $\{v_1, v_1 + v_2, v_1 + v_2 + v_3, v_1 + v_2 + v_3 + v_4\}$ is a linearly independent set.
- (iii) Find a basis for the subspace W of \mathbb{C}^4 .

$$W = \left\{ \begin{bmatrix} a+b-2c \\ a+b-2c+d \\ -2a+2b+4c-d \\ b+d \end{bmatrix} \mid a, b, c, d \in \mathbb{C} \right\}$$

[5+5+10=20]

$$(i) \alpha(v_1 - v_2) + \beta(v_2 - v_3) + \dots + \gamma(v_{n-1} - v_n) + \delta(v_n - v_1) = 0$$

clearly addition of all vectors yield 0;

$$\text{i.e. } (v_1 - v_2) + (v_2 - v_3) + \dots + (v_{n-1} - v_n) + (v_n - v_1) = 0$$

\therefore all the vectors can be put in form

$$\alpha_1(v_1 - v_2) + \alpha_2(v_2 - v_3) + \dots + \alpha_{n+1}(v_n - v_1) = 0$$

where each $\alpha_i \neq 0$. Hence it is linearly dependent set

(ii) Given v_1, v_2, v_3, v_4 is linearly independent set in \mathbb{C}^{35}

(iii) Given v_1, v_2, v_3, v_4 is linearly independent set in \mathbb{C}^{35}

$$\begin{aligned} & (v_1, v_1 + v_2, v_1 + v_2 + v_3, v_1 + v_2 + v_3 + v_4) \\ & = \alpha_1(v_1, v_2, v_3, v_4) + \alpha_2(0, 1, 1, 1) + \alpha_3(0, 0, 1, 1) + \alpha_4(0, 0, 0, 1) \\ & = v_1(1, 1, 1, 1) + v_2(0, 1, 1, 1) + v_3(0, 0, 1, 1) + v_4(0, 0, 0, 1) \end{aligned}$$

or $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ clearly it represents reduced row echelon form

however if each vector is considered as $\alpha_j + \beta_j(i)$;

$$\begin{aligned} & \text{then } = \alpha_1(1, 1, 1, 1) + \alpha_2(0, 1, 1, 1) + \alpha_3(0, 0, 1, 1) + \alpha_4(0, 0, 0, 1) \\ & \quad + i\{\beta_1(0, 1, 1, 1) + \beta_2(0, 0, 1, 1) + \beta_3(0, 0, 0, 1) + \beta_4(0, 0, 0, 1)\} \end{aligned}$$

clearly this represents linearly independent set ; as shown that both can separated in matrix represented.

$$(iii) W = \left\{ \begin{bmatrix} a+b-2c \\ ab-2c+d \\ -2a+2b+4c-d \\ b+d \end{bmatrix} \mid a, b, c, d \in \mathbb{C} \right\}$$

$x_i \in W$

$$x_i = a(1, 1, -2, 0) + b(1, 1, 2, 1) + c(-2, -2, 1, 0) + d(0, 1, 1, 1)$$

where $a, b, c, d \in \mathbb{C}$.
here each a, b, c, d is $a_r + i a_i$; where a_r is real and a_i is imaginary.

$$\sim \left(\begin{array}{cccc} 1 & 1 & -2 & 0 \\ 1 & 1 & 2 & 1 \\ -2 & -2 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array} \right) \xrightarrow{\substack{R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 + 2R_1}} \left(\begin{array}{cccc} 1 & 1 & -2 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \end{array} \right)$$

$$\sim \left(\begin{array}{cccc} 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \text{ i.e basis } \{ (1, 1, -2, 0), (0, 1, 1, 1), (0, 0, 4, 0) \} \text{ fb.}$$

or $(a, b, -2a, 0)$, $(0, b_1 - b, b)$,
 $(0, 0, 4b, 0)$ or we can put as

$$\left(\begin{array}{c} a \\ a+b \\ -2a-b+4c \\ b+c \end{array} \right) \sim \left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 4 \\ 1 \end{pmatrix} \right\} \text{ is required basis}$$

2. (b) By using Lagrange's multipliers method find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$. [14]

$$df = dx + 2dy + 3dz \quad (1); \text{ let } g: x - y + z - 1 = 0$$

$$dg = dx - dy + dz \quad (2) \text{ and } h: x^2 + y^2 - 1 = 0$$

$$dh = xdx + ydy \quad (3)$$

∴ by Lagrange's multiplier method:

$$dx + 2dy + 3dz + \lambda_1(dx - dy + dz) + \lambda_2(xdx + ydy) = 0$$

$$\text{or } 1 + \lambda_1 + x\lambda_2 = 0 \quad (1) \quad \text{& } 3 + \lambda_1 = 0$$

$$2 - \lambda_1 + \lambda_2 y = 0 \quad (2)$$

$$\text{i.e. } \lambda_1 = -3; \quad (1)$$

$$x\lambda_2 = 2 \quad (2)$$

$$3 + \lambda_1 = 0$$

$$y\lambda_2 = -5$$

$$x^2 + y^2 = 1 \Rightarrow \lambda_2 = \pm \sqrt{29}$$

$$x = \pm \frac{2}{\sqrt{29}}, \quad y = \pm \frac{5}{\sqrt{29}}; \quad z = 1 - \frac{7}{\sqrt{29}} \quad \text{or}$$

$$\left(\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}, 1 - \frac{7}{\sqrt{29}} \right) \text{ or } \left(\frac{-2}{\sqrt{29}}, \frac{5}{\sqrt{29}}, \frac{1+7}{\sqrt{29}} \right)$$

$$f = \left(\frac{-2}{\sqrt{29}} \right) + \frac{2(5)}{\sqrt{29}} + 3 \left(1 + \frac{7}{\sqrt{29}} \right) = \frac{\sqrt{29} + 3}{\sqrt{29}}$$

ANS

$$\text{at } (x_1, y_1, z_1) = \left(\frac{-2}{\sqrt{29}}, \frac{5}{\sqrt{29}}, 1 + \frac{7}{\sqrt{29}} \right)$$

$$df_x = 1 + 2\frac{dy}{dx} + 3\frac{dz}{dx},$$

$$= 1 + 5\frac{dy}{dx} - 3$$

$$f_{xx} = 5\frac{d^2y}{dx^2} < 0; \text{ hence } f \text{ is max}$$

$$\left\{ \begin{array}{l} \frac{dz}{dx} = \frac{dy}{dx} - 1 \\ \frac{dy}{dx} = \frac{x}{y} \end{array} \right\} \text{ from (2)}$$

$$\left\{ \frac{dy}{dx} = \frac{x}{y} \right\} \rightarrow$$

$$\frac{dy}{dx} = \frac{-1(1+x^2)}{y^2} < 0$$

2. (c) (i) The plane $x - 2y + 3z = 0$ is rotated through a right angle about its line of intersection with the plane $2x + 3y - 4z - 5 = 0$. Find the equation of the plane in its new position.

- (ii) A variable plane is parallel to the given plane $(x/a) + (y/b) + (z/c) = 0$ and meets the axes in A, B, C respectively. Prove that the circle ABC lies on the curve

$$yz\left(\frac{b+c}{c}\right) + zx\left(\frac{a+c}{c}\right) + xy\left(\frac{a+b}{a}\right) = 0$$

[16]

(ii) $A' = (a, 0, 0)$, $B' = (0, b, 0)$, $C' = (0, 0, c)$

\therefore the plane $\boxed{\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = \lambda}$ ①

$\therefore A = (\lambda a, 0, 0)$, $B = (0, b\lambda, 0)$, $C = (0, 0, c\lambda)$

Given variable point ; where the plane meets axes

circle $OABC$:

$$u^2 + v^2 + w^2 + 2uv + 2uw + 2vw = 0 \quad \text{where } u = \frac{x}{a}, v = \frac{y}{b}, w = \frac{z}{c}$$

in it passes through

$$\therefore (a\lambda)^2 + u(a\lambda)^2 = 0$$

$$\therefore u = -a\lambda/2$$

$$\text{likewise } v = -b\lambda/2$$

likewise $v = -b\lambda/2$

Putting u, v, w back in ②; we get

$$x^2 + y^2 + z^2 - a\lambda x - b\lambda y - c\lambda z = 0 \quad \text{②}$$

to find the required curve ; we observe from

①. λ is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0$ and

$$-x^2 - y^2 - z^2 - \lambda(ax + by + cz)\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) = 0$$

$$\therefore -x^2 - y^2 - z^2 - \lambda(ax + by + cz)\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) = 0$$

$$\text{which is } xy\left(\frac{a}{b} + \frac{b}{a}\right) + yz\left(\frac{b}{c} + \frac{c}{b}\right) + zx\left(\frac{a}{c} + \frac{c}{a}\right) = 0$$

hence proved

$$(i) P_1: x - 2y + 3z = 0 \quad q) P_2: 2x + 3y - 4z - 5 = 0$$

Any plane is $P_1 + \lambda P_2 = 0$; through intersection of P_1 & P_2

$$\text{or } (x(1+2\lambda)) + y(-2+3\lambda) + z(3-4\lambda) + (-5\lambda) = 0$$

and if this is plane $\perp r$ to $\vec{0}$ ~~exist~~

i.e. d.r.s of P_1 is $1, -2, 3$

$$\therefore (1+2\lambda) + (-2+3\lambda)(-2) + (3-4\lambda)(3) = 0$$

$$1+4+9+\lambda(2-6-12) = 0 \quad \text{or} \quad \lambda = 7/8$$

$$\therefore x(1+2(7/8)) + y(3 \times \frac{7}{8} - 2) + z(\frac{3-4 \times 7}{8}) - 5 \times \frac{7}{8} = 0$$

$$\boxed{22x + 5y - 4z = 35}$$

is the required new rotated plane equation

3. (a) (i) Find a basis for $\langle S \rangle$, where

$$S = \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \\ 3 \end{bmatrix} \right\}$$

- (ii) Find the values of λ for which the equations

$$(\lambda - 1)x + (3\lambda + 1)y + 2\lambda z = 0$$

$$(\lambda - 1)x + (4\lambda - 2)y + (\lambda + 3)z = 0$$

$$2x + (3\lambda + 1)y + 3(\lambda - 1)z = 0$$

are consistent, and find the ratios of $x : y : z$ when λ has the smallest of these values. What happens when λ has the greater of these values. [5+13=18]

4. (a) (i) Define $T : \mathbb{C}^2$ to \mathbb{C}^2 by $T([z_1, z_2]) = (iz_1, (1+i)z_2 - z_1)$. Let \mathbb{C}^2 have the basis $S = \{(i, 0), (0, 1)\}$. Calculate M_T .
- (ii) If A is a non-singular matrix, then show that
- $$\text{adj } \text{adj } A = |A|^{n-2} A.$$

(iii) Using Cayley-Hamilton theorem, find A^8 , if $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

$$(a)(i) \quad T((z_1, z_2)) = (iz_1, (1+i)z_2 - z_1) \quad [18]$$

$$T(0, 1) = (0, (1+i)) = (0, 1) + (0, i)$$

$$T(-1, 0) = (-1, i) = (-1, 0) + (0, i)$$

$$T(0, 1) = 0 \cdot (i, 0) + (1+i) (0, 1)$$

$$T(i, 0) = i \cdot (i, 0) + i (0, 1)$$

$$\therefore M_T = \begin{bmatrix} i & 0 \\ i & 1+i \end{bmatrix} \text{ is required} \quad M_T$$

(ii) we know that ; $|A| \neq 0$.

$$A \cdot \text{Adj } A = |A| I_n \quad \cancel{(A)}$$

$$|A| \cdot |\text{Adj } A| = |A|^n$$

$$\text{or } |\text{Adj } A| = (A)^{n-1}$$

$$\text{Also : } (\text{Adj } A)(\text{Adj } \text{Adj } A) = |(\text{Adj } A)| I_n$$

multiplying by A { By putting $\text{Adj } A$ instead of A^n }

$$A(\text{Adj } A)(\text{Adj } \text{Adj } (A)) = |(\text{Adj } A)|^2 A$$

$$|A| \cdot (\text{Adj } \text{Adj } (A)) = |A|^{n-1} \cdot A$$

$$\text{or } \boxed{(\text{Adj } \text{Adj } (A)) = |A|^{n-2} \cdot A} \quad \text{here proved}$$

~~X6~~

(iii) $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}; |A| \neq 0$.

$$|(A - \lambda I)| = \begin{bmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{bmatrix} = 0 \quad \text{or} \quad \lambda^2 - 1 - 4 = 0$$

$$\boxed{A^2 - 5I = 0}$$

{ by Cayley's Hamilton theorem;
a every characteristic equation is satisfied by its
own matrix }

$$A^2 = 5I;$$

$$A^4 = 25I$$

$$A^6 = 125I$$

or

$$\boxed{A^8 = 625I}$$

$$\text{or } A^8 = \begin{bmatrix} 625 & 0 \\ 0 & 625 \end{bmatrix}$$

4. (b) (i) Let $z = f(t)$, $t = \frac{x+y}{xy}$. Show that $x^2 \frac{\partial z}{\partial x} = y^2 \frac{\partial z}{\partial y}$.

(ii) Evaluate $\iiint z \, dx \, dy \, dz$ over the volume enclosed between the cone $x^2 + y^2 = z^2$

and the sphere $x^2 + y^2 + z^2 = 1$ on positive side of xy-plane. [16]

~~-06-~~

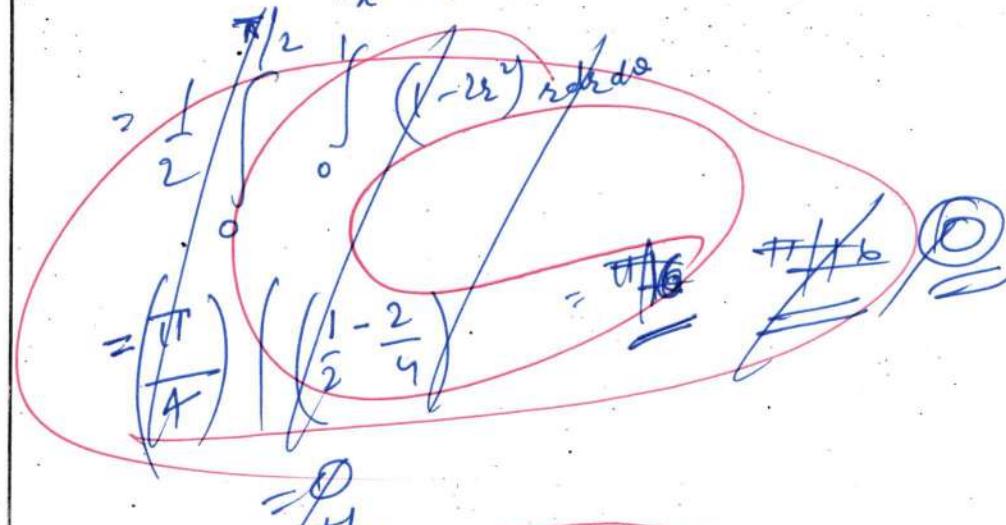
(ii) $\iiint z \, dx \, dy \, dz$

$$\begin{aligned} & \left[z = \frac{1}{\sqrt{2}} \right] ; \text{ point of intersection of cone and sphere} \\ & = \int_0^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_0^{\sqrt{1-y^2-z^2}} z \, dx \, dy \, dz \end{aligned}$$

$$= \frac{1}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} (1 - 2(x^2 + r^2)) dx dy$$

put $dx dy = r dr d\theta$
 $x^2 + y^2 = r^2$

} changing to
polar
coordinates



$$= \frac{1}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} (1-r^2) r dr d\theta$$

$$= \frac{1}{2} \left(\frac{\pi}{2}\right) \left(\frac{1}{2} - \frac{1}{4} \right) = \cancel{\frac{\pi}{16}}$$

(ii) $z = f(t)$, $t = \frac{1}{x} + \frac{1}{y}$

$$\frac{\partial t}{\partial x} = -\frac{1}{x^2}, \quad ; \quad \frac{\partial t}{\partial y} = -\frac{1}{y^2}$$

$$\frac{\partial z}{\partial x} = f'(t) \cdot \frac{\partial t}{\partial x} \quad \& \quad \frac{\partial z}{\partial y} = f'(t) \cdot \frac{\partial t}{\partial y}$$

$$\frac{\partial z}{\partial x} = f'(t) \cdot -\frac{1}{x^2} \quad \& \quad \frac{\partial z}{\partial y} = f'(t) \cdot -\frac{1}{y^2}$$

$$\Rightarrow \left(\frac{\partial z}{\partial x}\right)(-x^2) = \left(\frac{\partial z}{\partial y}\right)(-y^2)$$

hence
proved

4. (c) CP, CQ are any two conjugate semi-diameters of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$,
 $z = c$, CP', CQ' are the conjugate diameters of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$,
 $z = -c$ drawn in the same directions as CP and CQ, Prove that the hyperboloid
 $(2x^2/a^2) + (2y^2/b^2) - (z^2/c^2) = 1$ is generated by either PQ' or P' Q'. [16]

Q. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z = c; P = (a\cos\theta, b\sin\theta, c)$
 $Q = (-a\sin\theta, b\cos\theta, c)$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z = -c; P' = (a\cos\theta, b\sin\theta, -c)$
 $Q' = (-a\sin\theta, b\cos\theta, -c)$

~~-14-~~

Line PQ': $\frac{x - a\cos\theta}{a(\cos\theta + \sin\theta)} = \frac{y - b\sin\theta}{b(\sin\theta - \cos\theta)} = \frac{z - c}{2c} = r_1$

so $\left(\frac{x}{a}\right) = \cos\theta + r_1(\cos\theta + \sin\theta); z = (2r_1 + 1)c$
 $\left(\frac{y}{b}\right) = \sin\theta + r_1(\sin\theta - \cos\theta), \quad \frac{z}{c} = 2r_1 + 1$
 $\frac{z^2}{c^2} = 4r_1^2 + 4r_1 + 1$

$$\begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{b^2} &= \frac{\cos^2\theta + r_1^2(\sin^2\theta + \cos^2\theta + 2\cos\theta \cdot \sin\theta) +}{\sin^2\theta + r_1^2(\sin^2\theta - \cos^2\theta + 2\cos\theta \cdot \sin\theta)} + \\ &\quad + 2r_1 \cos\theta (\cos\theta + \sin\theta) + 2r_1 \sin\theta (\sin\theta - \cos\theta) \\ &= 2r_1^2 + 2r_1 + 1 \end{aligned}$$

or $\frac{2x^2}{a^2} + \frac{2y^2}{b^2} = 2(2r_1 + 1)^2 + 1$
as $\left(\frac{z}{c}\right)^2 = (2r_1 + 1)^2$

$$\boxed{\frac{2x^2}{a^2} + \frac{2y^2}{b^2} - \frac{z^2}{c^2} = 1}$$

Likewise proceeding similarly, for P'Q'

$$\frac{x - a \cos \theta}{a(\cos \theta + \sin \theta)} = \frac{y - b \sin \theta}{b(\sin \theta - \cos \theta)} = \frac{z + c}{-2c} = z_2$$

This equation is similar to PQ'.

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = 2z_2^2 + 2z_2 + 1$$

$$\text{or } \left(\frac{z}{c}\right)^2 - \left[-(2z_2 + 1)\right]^2 = 4z_2^2 + 4z_2 + 1$$

$$\therefore \boxed{\frac{2x^2}{a^2} + \frac{2y^2}{b^2} - \frac{z^2}{c^2} = 1} \quad \text{hence proved}$$

SECTION - B

5. (a) (i) Solve : $x \cos(y/x)(y dx + x dy) = y \sin(y/x)(x dy - y dx)$
(ii) Solve $y(x^2y^2 + 2) dx + x(2 - 2x^2y^2) dy = 0$

[10]

$$(i) \cancel{x(x^2+y^2) \, d(xy)} = \left(\frac{y}{x}\right) \tan(y/x) \cdot \left[\left(d(y/x)\right)x^2\right]$$

$$\text{or } \frac{d(xy)}{xy} = \tan(y/x) \cdot d(y/x)$$

$$\log(xy) = \log \sec(y/x) + C$$

~~or~~ or $\boxed{xy / \sec(y/x) = C'}$

$$(ii) y dx(x^2y^2 + 2) + x dy(2 - 2x^2y^2) = 0$$

is in form $y dx f_1(xy) + x dy f_2(xy) = 0$

integrating factor is $\frac{1}{Mx - Ny} = \frac{1}{xy(3x^2y^2)}$

$$\text{or } I.F = \frac{1}{3x^3y^3}$$

$$\frac{ydx}{3x^3y^3} \left(x^2y^2 + 2 \right) + \frac{x dy}{3x^3y^3} (2 - 2x^2y^2) > 0$$

$$\frac{2(ydx + xdy)}{3(xy)^3} + \frac{dx}{3x} - \frac{2dy}{3y} = 0$$

clearly it's in exact form

$$\frac{2(d(xy))}{3(xy)^3} + \frac{dx}{3x} - \frac{2dy}{3y} = 0$$

$$\boxed{\frac{-1}{3x^2y^2} + \frac{\log x}{3} - \frac{2\log y}{3} = C}$$

5. (b) Solve $(px^2 + y^2)(px + y) = (p + 1)^2$ by reducing it to Clairaut's form and find its singular solution. [10]

$$\text{let } v = xy \quad \text{and } u = x + y ; \frac{dv}{du} = \frac{px+y}{p+1} = p'$$

$$px+y = pp' + p' \Rightarrow p(x-p') = p'-y \Rightarrow$$

$$p = \frac{p'y}{x-p'}$$

$$\left(\frac{p'-y}{x-p'} \cdot x^2 + y^2 \cdot \frac{(x-p')}{(x-p')} \right) p' = \frac{(x-y)}{(x-p')}$$

$$\text{or } p(x^2 - y^2) + xy(y-x) = (x-y)$$

$$(p^2(x+y) - xy(p)) = 1$$

$$\text{or } p'^2(x+y) - 1 = xy p'$$

$$v = p'u - \frac{1}{p'} \quad \text{is the reduced equation from Clairaut's form}$$

∴ the singular soln is obtained by putting $p' = c$

$$\therefore v = cu - \frac{1}{c}$$

$$xy = c(x+y) - \frac{1}{c}$$

$$c^2(x+y) - cxy + 1 = 0$$

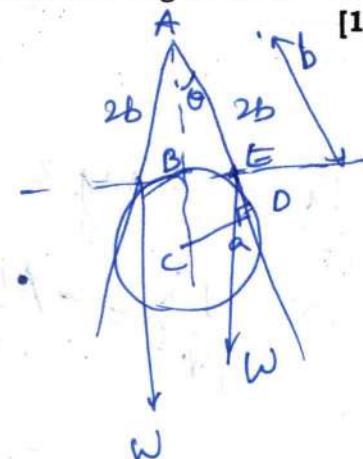
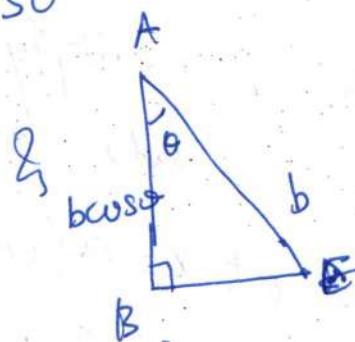
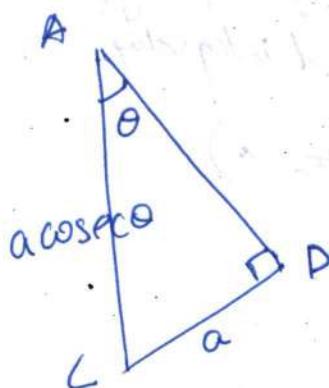
$$x^2y - 4xy^2 + 4y^3 = 0 \quad \text{is reqd singular solution}$$

$$B^2 - 4AC = 0 \Rightarrow x^2y - 4y(x+y)^2 = 0 \quad \text{and} \quad 2xy + 2(x+y) = 0$$

5. (c) Two equal rods, AB and AC, each of length $2b$, are freely jointed at A and rest on a smooth vertical circle of radius a . Show that if 2θ be the angle between them then $b \sin^3 \theta = a \cos \theta$. [10]

$$AC = a \cosec \theta$$

$$BC = b \cos \theta$$



If the rod rests in equilibrium; by principle of virtual work

$2w \delta \cdot (BC) = 0$; where w is weight of
(BC) is distance of C(G) of rod from fixed centre
of virtual

$$\delta \cdot S(BC) = 0$$

$$S \cdot (AC - AB) = 0$$

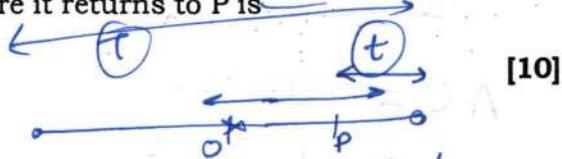
$$\cancel{S \cdot (a \cosec \theta - b \cos \theta) = 0}$$

$$\cancel{- a \cosec^2 \theta \cdot \cos \theta + b \sin \theta = 0}$$

$\therefore \cancel{a \cos \theta = b \sin^3 \theta}$ hence proved

5. (d) A particle is performing a simple harmonic motion of period T about a centre O and it passes through a point P where $OP = b$ with velocity v in the direction OP; prove that the time which elapses before it returns to P is

$$\frac{T}{\pi} \tan^{-1} \left(\frac{vT}{2\pi b} \right).$$



[10]

$$m \frac{d^2x}{dt^2} \ddot{x} - mg x$$

$$\text{or } \frac{d^2x}{dt^2} = -\mu x$$

; multiplying by $\frac{dx}{dt}$ both sides and integrating

$$\left(\frac{dx}{dt} \right)^2 = -\mu x^2 + C; \text{ at } x=a \text{ (extreme)}$$

here at $x=b$;

$$\frac{dx}{dt} = v$$

$$\text{or } C = V^2 + \mu b^2$$

$$V=0$$

$$V^2 = -\mu b^2 + C$$

$$= 1 \text{ or}$$

$$\frac{dx}{dt} = \sqrt{V^2 + \mu(b^2 - x^2)}$$

$$= -dt \frac{\sin x}{\sqrt{\mu}}$$

$$\frac{dx}{dt} = \sqrt{V^2 + \mu b^2 - \mu x^2}$$

$$\text{at } x=a, V=0 \Rightarrow \frac{dx}{dt} = -dt$$

$$\frac{1}{\sqrt{\mu}} \sin \frac{x}{a} \Big|_0^a = (T) \Rightarrow T = \frac{\pi}{2} \sqrt{\frac{a^2}{\mu}}$$

$$\text{Also: } \frac{V^2}{\mu} = \sqrt{a^2 - b^2} \\ \frac{V^2}{\mu} = \frac{a^2 - b^2}{\mu} = \frac{2\pi}{T}$$

$$\text{total time period } T = \frac{2\pi}{\sqrt{\mu}}$$

and for top to P is;

$$\frac{1}{\sqrt{\mu}} \sin \frac{x}{a} \Big|_b^a = \left(\frac{t}{T} \right) \text{ or } t = \frac{2}{\sqrt{\mu}} \cos \frac{a}{\sqrt{\mu}} \frac{2 \tan^{-1} \frac{V}{\sqrt{\mu} b}}{\sqrt{\mu}} = \frac{2 \tan^{-1} \frac{V}{\sqrt{\mu} b}}{\sqrt{\mu}}$$

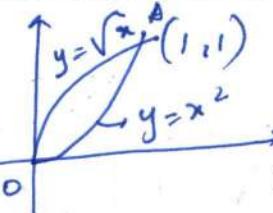
$$\text{as } \cos^2 b/a^2 \tan^2 \frac{V^2 - b^2}{a^2} = \frac{V^2}{\mu b^2}$$

5. (e) Verify Green's theorem in the plane for

$$\oint_C [(3x^2 - 8y^2)dx + (4y - 6xy)dy],$$

where C is the boundary of the region defined by

$$y = \sqrt{x}, y = x^2.$$



[10]

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy; \quad \frac{\partial N}{\partial x} = -6y \\ \frac{\partial M}{\partial y} = -16x$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} 10y dxdy$$

$$\therefore \int_0^1 y^2 \Big|_{x^2}^{\sqrt{x}} dx = \int_0^1 (x - x^4) dx = \frac{3}{2}$$

$$\int_0^1 \left[(3x^2 - 8(x^2)^2) dx + (4x^2 - 6x^3) 2x dx \right] \\ = \int_0^1 dx (3x^2 - 8x^4 + 8x^3 - 12x^4) = \int_0^1 dx \left(\frac{3x^2 + 8x^3}{-20x^4} \right)$$

$$= -\frac{1}{2}$$

$$\int_0^1 (3x^2 - 8x) dx + (4\sqrt{x} - 6x\sqrt{x}) \frac{dx}{2\sqrt{x}} \\ = \int_0^1 (3x^2 - 14x + 4) dx = (-1 + 7 - 4) \\ = \int_0^1 (3x^2 - 11x + 2) dx = -1 + \frac{11}{2} - 2 = \frac{5}{2}$$

$$\text{Adding } ① \text{ & } ②: \quad \oint_{y=\sqrt{x}} M dx + N dy = \frac{3}{2} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

- (a) (i) Evaluate $L^{-1}\{e^{4-3s} / (s+4)^{5/2}\}$
(ii) By using Laplace transform solve $(D^2 + m^2)x = a \sin nt$, $t > 0$ where x , Dx equal to x_0 and x_1 , when $t = 0$, $n \neq m$. [5+13=18]

$$(i) e^{4t} L^{-1} \left\{ \frac{e^{-3s}}{(s+4)^{5/2}} \right\}$$

$$L^{-1} \left\{ \frac{1}{(s+4)^{5/2}} \right\} = e^{-4t} L^{-1} \left\{ \frac{1}{s^{5/2}} \right\} = \frac{e^{-4t}}{\sqrt{s} t^{3/2}}$$

$$L^{-1} \left\{ \frac{e^{-3s}}{(s+4)^{5/2}} \right\} = e^{-4(t-3)} \cdot \frac{e^{-4(t-3)}}{\sqrt{s} t^{3/2}} = e^{4-4t+12} \cdot \frac{(t-3)^{3/2}}{\sqrt{s} t^{3/2}} ; t > 3/2$$

$$L^{-1} \left\{ \frac{e^{4-3s}}{(s+4)^{5/2}} \right\} = e^{4-4t+12} \cdot \frac{(t-3)^{3/2}}{\sqrt{s} t^{3/2}} ; t < 3/2$$

$$\left\{ L^{-1}\{e^{-ap}/s\} = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases} \right.$$

heaviside step function

$$(ii) (D^2 + m^2)x = a \sin nt$$

Taking Laplace on both sides

$$s^2 F(s) - sF(0) - f'(0) = \frac{an}{s^2 + n^2}$$

$$s^2 F(s) - SF(0) - f'(0) + m^2 F(s)$$

$$(F)(s^2 + m^2) = x_0 s + x_1 + \frac{an}{s^2 + n^2}$$

$$F(s) = \frac{x_0 \cdot s}{s^2 + m^2} + \frac{x_1}{s^2 + m^2} + \frac{an}{(n^2 - m^2)} \left(\frac{1}{s^2 + m^2} - \frac{1}{s^2 + n^2} \right)$$

$$x_2 F(t) = x_0 \cos mt + x_1 \sin mt + \frac{an}{(n^2 - m^2)} \left(\frac{\sin mt - \sin nt}{m} \right)$$

6. (c) (i) Find the curvature K , and the torsion τ for the space curve $x = t - t^3/3$, $y = t^2$, $z = t + t^3/3$.

(ii) If $A = 5t^2 \mathbf{i} + t\mathbf{j} - t^3 \mathbf{k}$ and $B = \sin t\mathbf{i} - \cos t\mathbf{j}$, find $\frac{d}{dt}(A \times B)$, $\frac{d}{dt}(A \times B)$ and $\frac{d}{dt}(A \times A)$

$$(i) \vec{r} = \left(t - \frac{t^3}{3}\right)\mathbf{i} + t^2\mathbf{j} + \left(t + \frac{t^3}{3}\right)\mathbf{k} \quad [18]$$

$$\left(\frac{dr}{dt}\right) = (1-t^2)\mathbf{i} + 2t\mathbf{j} + (1+t^2)\mathbf{k}$$

$$\frac{d^2r}{dt^2} = -2t\mathbf{i} + 2\mathbf{j} + 2t\mathbf{k}$$

$$\frac{d^3r}{dt^3} = -2\mathbf{i} + 0\mathbf{j} + 2\mathbf{k}$$

$$\left(\frac{dr}{dt} + \frac{d^2r}{dt^2}\right) \cdot \left(\frac{dr}{dt}\right)^3 = x$$

$$\frac{de}{dt} \times \frac{d^2r}{dt^2} = 2(t^2-1)\mathbf{i} + 4t\mathbf{j} + 2(t^2+1)\mathbf{k}$$

$$\left| \frac{dr}{dt} \times \frac{d^2r}{dt^2} \right|^2 = 2$$

$$(t^2-1)^2 + (2t)^2 + (t^2+1)^2$$

$$\left| \frac{d^2r}{dt^2} \right| = \left| \frac{ds}{dt} \right| =$$

$$\sqrt{(1-t^2)^2 + (2t)^2 + (1+t^2)^2}$$

$$x = \frac{2}{(1-t^2)^2 + (2t)^2 + (1+t^2)^2} = \frac{2}{(2+2t^4)+4t^2}$$

$$= \underline{\underline{\left(\frac{1}{t^2+1}\right)^2}}$$

$$i \quad \tau = \frac{dr}{dt} \cdot \left(\frac{dr}{dt} \times \frac{d^2r}{dt^2} \right)$$

$$\text{or } \tau = \left(\frac{dr}{dt} \times \frac{d^2r}{dt^2} \right) \cdot \left(\frac{d^3r}{dt^3} \right) = \underline{\underline{8/x^2}}$$

$$\begin{bmatrix} \mathbf{i} & -\mathbf{j} & \mathbf{k} \\ 1-t^2 & 2t & 1+t^2 \\ -2t & 2 & 2t \end{bmatrix}$$

$$z = \underline{8(t^2+1)^7}$$

(ii)

7. (a) Solve $(x^2 D^2 - xD + 1)y = (\log x \sin \log x + 1)/x$. [15]

let $\log x = z$; then by euler's formula: $\frac{d}{dx} = D$

$$x^2 D^2 = D(D-1), \text{ & } xD = D$$

$$\text{or: } (D(D-1) + D + 1)Y = \left(\frac{z \sin z + 1}{e^z} \right)$$

$$\text{or } (D^2 - 2D + 1)Y = e^z(z \sin z + 1)$$

$$\text{for R.F. } (D^2 - 2D + 1) = 0 \Rightarrow m^2 - 2m + 1 = 0$$

$$\text{or } Y = (C_1 + C_2 z) e^z$$

$$P.I. = e^{-z} \cdot (z \sin z + 1)$$

$$= e^{-z} \frac{1}{(D-1)^2} (z \sin z + 1) \quad \left\{ \begin{array}{l} \text{using } \\ \frac{f(e^{az})g(z)}{f(D)} = e^{\frac{az}{D}} \frac{1}{f(D+a)} \end{array} \right.$$

$$= e^{-z} \frac{1}{(D-1)^2}$$

$$= e^{-z} \frac{1}{(D-2)^2} (z \sin z + 1)$$

$$(P.I.)_1 = \frac{1}{(D-2)^2} (z \sin z + 1)$$

$$\left\{ \text{using } \frac{1}{f(D)} \cdot xV = \left(x \frac{1}{f(D)} V - \frac{f'(D)}{(f(D))^2} \cdot V \right) \right\}$$

$$= z \frac{1}{(D-2)^2} \sin z - \frac{2(D-2)}{(D-2)^4} \cdot \sin z$$

$$= z \cdot \frac{1}{(D-2)^2} \sin z - \frac{2}{(D-2)^3} \cdot \sin z$$

$$\begin{aligned}
 &= z \cdot \frac{1}{D^2 - 4D + 4} \sin z - \frac{2}{(D^3 - 6D^2 + 12D - 8)} \cdot \sin z \\
 &= z \cdot \frac{(3D + 4D)}{-16(D^2 + 1)} \sin z - \frac{2}{11D - 2} \cdot \sin z \\
 &= \frac{z(3\sin z + 4\cos z)}{25} + \frac{2(11D + 2)}{125} \sin z \\
 &= \frac{z(3\sin z + 4\cos z)}{25} + \frac{2\cos z + 4\sin z}{125}
 \end{aligned}$$

Also $\frac{1}{(D-2)^2} (1) = \frac{t+1}{x} - \frac{1}{4}$

\therefore final soln $\therefore (F + PI)$
 or $(C_1 + C_2 \log x) x + \frac{1}{x} \left\{ \log x \left(\frac{3\sin(\log x) + 4\cos(\log x)}{25} + \frac{1}{4} \right. \right.$
 $\left. \left. + \frac{2\cos(\log x) + 4\sin(\log x)}{125} \right) \right\}$

7. (b) A particle starts from rest at the cusp of a smooth cycloid whose axis is vertical and vertex downwards. Prove that when it has fallen through half the distance measured along the arc to the vertex, two-thirds of the time of descent will have elapsed. [17]

The equation of motion

$$R - mg \cos \psi = \frac{mv^2}{r}$$

$$-mg \sin \psi = \frac{md\dot{\psi}}{dt}$$

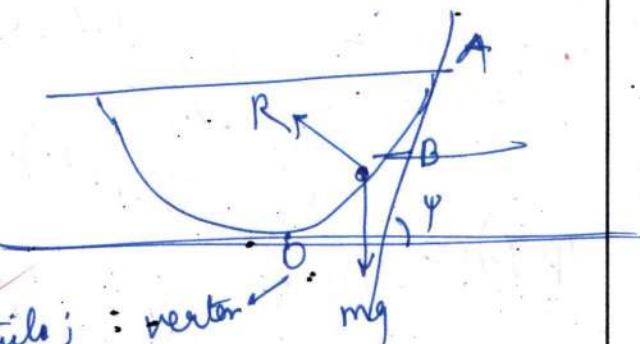
where m is mass of particle; R reaction at any time

R is reaction at any time

$$\text{Also } s = \frac{1}{2} a \sin \psi$$

$$\begin{aligned}
 \left(\frac{ds}{dt} \right)^2 &= -g \cdot s^2 + A \\
 \left(\frac{ds}{dt} \right)^2 &= \frac{g s / 4a}{dt^2} = \frac{ad\dot{s}}{dt^2}
 \end{aligned}$$

after multiplying by $\frac{2ds}{dt}$, on both sides and then integrating



at $t=0$; $v=0 = \frac{ds}{dt}$

$$\therefore S = 4a \quad \text{at } t=0.$$

$$A = +4a g$$

$$\left(\frac{ds}{dt}\right)^2 = \frac{g}{4a} ((4a)^2 - s^2)$$

$$\text{or } \frac{ds}{dt} = -\sqrt{\frac{g}{4a}} \sqrt{4a^2 - s^2} \quad \left. \begin{array}{l} \text{- sign indicates, the} \\ \text{s is decreasing} \end{array} \right\}$$

total time of descent from $s = 4a$ to $s = 0$

$$ds = -\sqrt{\frac{g}{4a}} dt$$

$$\int_0^{4a} \frac{1}{\sqrt{(4a)^2 - s^2}} ds$$

$$\text{or } \sin^{-1} \left(\frac{s}{4a} \right) \Big|_0^{4a} = \sqrt{\frac{g}{4a}} \cdot t_t$$

$$t_t = \frac{\pi}{2} \sqrt{\frac{4a}{g}}$$

①

if it has come down for half distance; then
 $s = 2a$; ie time taken to travel from $s = 4a$ to $2a$.

$$\int_{2a}^{4a} \frac{1}{\sqrt{(4a)^2 - s^2}} ds = -\sqrt{\frac{g}{4a}} \cdot t_{1/2}$$

$$t_{1/2} = \sqrt{\frac{4a}{g}} \sin^{-1} \left(\frac{s}{4a} \right) \Big|_{2a}^{4a} = \left(\frac{\pi}{2} - \frac{\pi}{6} \right) \sqrt{\frac{4a}{g}}$$

$$t_{1/2} = \frac{\pi}{3} \sqrt{\frac{4a}{g}}$$

from ① & ② clearly $t_{1/2} = \frac{2}{3} t_t$; ie time taken to travel half is $\frac{2}{3}$ rd of total time of descent

7. (c) (i) Find the values of the constants a , b , c so that the directional derivative of $\phi = ax^2 + by^2 + cz^2$ at $(1, 1, 2)$ has a maximum magnitude 4 in the direction parallel to y -axis.
- (ii) Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$, and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$.
- (iii) Evaluate $\iint_S (\nabla \times F) \cdot dS$, where

$F = (x^2 + y - 4) \mathbf{i} + 3xy \mathbf{j} + (2xz + z^2) \mathbf{k}$ and S is the surface of the paraboloid $z = 4 - (x^2 + y^2)$ above the xy -plane. [4+4+12=20]

(i) $\nabla \phi$ is directional derivative

$$\nabla \phi = 2ax\mathbf{i} + 2by\mathbf{j} + 2cz\mathbf{k} \quad \text{at } (1, 1, 2) \text{ is}$$

$$\nabla \phi = 2a\mathbf{i} + 2b\mathbf{j} + 4c\mathbf{k}, \text{ also if this is}$$

parallel to y -axis means

and for magnitude 4, we have $4 = 2b \Rightarrow b = 2$

$\therefore a = 0, b = 2, c > 0$

(ii) $(\nabla \phi_1)$ and $(\nabla \phi_2)$

$$\phi_1: x^2 + y^2 + z^2 - 9 = 0 \Rightarrow (\nabla \phi_1) = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{(2, 1, 2)} = 4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$$

$$\phi_2: x^2 + y^2 - z - 3 = 0 \Rightarrow (\nabla \phi_2) = \frac{2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}}{(2, -1, 2)} = 4\mathbf{i} - 2\mathbf{j} - \mathbf{k}$$

Angle b/w $\nabla \phi_1$ & $\nabla \phi_2$ is

$$\cos \theta = \frac{(\nabla \phi_1) \cdot (\nabla \phi_2)}{|\nabla \phi_1| |\nabla \phi_2|} = \frac{16 + 4 - 4}{\sqrt{36} \sqrt{21}} = \frac{16}{3\sqrt{21}}$$

$$= \frac{16}{3\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

(m) At $z=0$ of paraboloid: $x^2 + y^2 = 4$

$$\nabla \times F = \begin{bmatrix} i & -j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y - 4 & 3xy & 2x^2 + 2 \end{bmatrix}$$

$$= -2z \hat{i} + (3y - 1) \hat{k}$$

as $z > 0$:

$$\nabla \times \vec{F} = +(3y - 1) \hat{k}$$

further by Gauss divergence theorem:

$$\iint_S G \cdot dS = \iiint_V (\nabla \cdot \vec{F}) dV$$

$$\iint_S (\nabla \times \vec{F}) \cdot n dS = \iiint_V \nabla \cdot (\nabla \times \vec{F}) dV = 0$$

$\left\{ \text{div} \cdot \text{curl } F = 0 \right\}$

$S_1 + S_2$

$$\iint_{S_1} (\nabla \times \vec{F}) \cdot n dS = - \iint_{S_2} (\nabla \times \vec{F}) \cdot n dS ; \quad \text{dvs for } S_1: \quad \begin{aligned} n &= -\hat{k} \\ &\{ \text{from fig} \} \end{aligned}$$

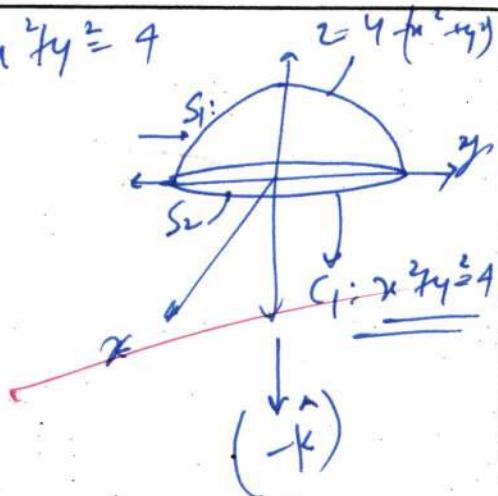
$$= \iint_{S_2} (3y - 1) dx dy$$

using polar co-ordinate

$$= \int_0^{2\pi} \int_0^2 [(3r \sin \theta) - 1] r dr d\theta \quad \left\{ \int_0^{2\pi} \int_0^r \sin \theta d\theta = 0 \right\}$$

$$= - \int_0^{2\pi} \int_0^2 -r dr d\theta$$

$$\boxed{\iint_S (\nabla \times \vec{F}) \cdot n dS = -4\pi}$$



8. (c) Verify Stokes theorem for $\mathbf{F} = xz \mathbf{i} - y \mathbf{j} + x^2y \mathbf{k}$, where S is the surface of the region bounded by $x = 0, y = 0, z = 0, 2x + y + 2z = 8$ which is not included in the xz plane. [17]

$$\iint (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \oint \mathbf{F} \cdot d\mathbf{r}$$

$$\nabla \times \mathbf{F} = x^2 \mathbf{i} - (2xy - x) \mathbf{j}$$

$$\mathbf{n} = (2i + j + k)^{-1}$$

$$\left[\begin{array}{ccc} i & -j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & -2xy & x \end{array} \right]$$

$$\iint (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \iint (2x^2 - 2xy + x) (dS)$$

$$= \iint \{2x^2 - 2x(8 - 2x - 2z)\} dx dz$$

$$= \iint \{6x^2 - 15x^2 + 2xz^2\} dx dz$$

$$= \int_0^4 \int_0^{4-x} (6x^2 - 15x^2 + 2xz^2) dx dz$$

$$= \int_0^4 \int_0^{4-x} (6x^2 - 15x^2 + 2xz^2) dx dz$$

$$= \int_0^4 \left[2x^3 - 6x^3 - 60x + 15x^2 + 32 + 2x^3 - 16x^2 \right] dx$$

$$= \int_0^4 (-4x^3 + 23x^2 - 60x + 32) dx$$

$$= -4x^4 + 23x^3 - 60x^2 + 32x$$

$$= -4x^4 + 23 \cdot 4^3 - 60 \cdot 4^2 + 32 \cdot 4$$

$$= -4^4 + 23 \cdot 4^3 - 60 \cdot 4^2 + 32 \cdot 4$$

$$= -256 + 23 \cdot 64 - 60 \cdot 16 + 128$$

$$= -256 + 1472 - 960 + 128$$

$$= 184$$

$$= 184 \pi$$

$$= 184 \pi \cos 90^\circ$$

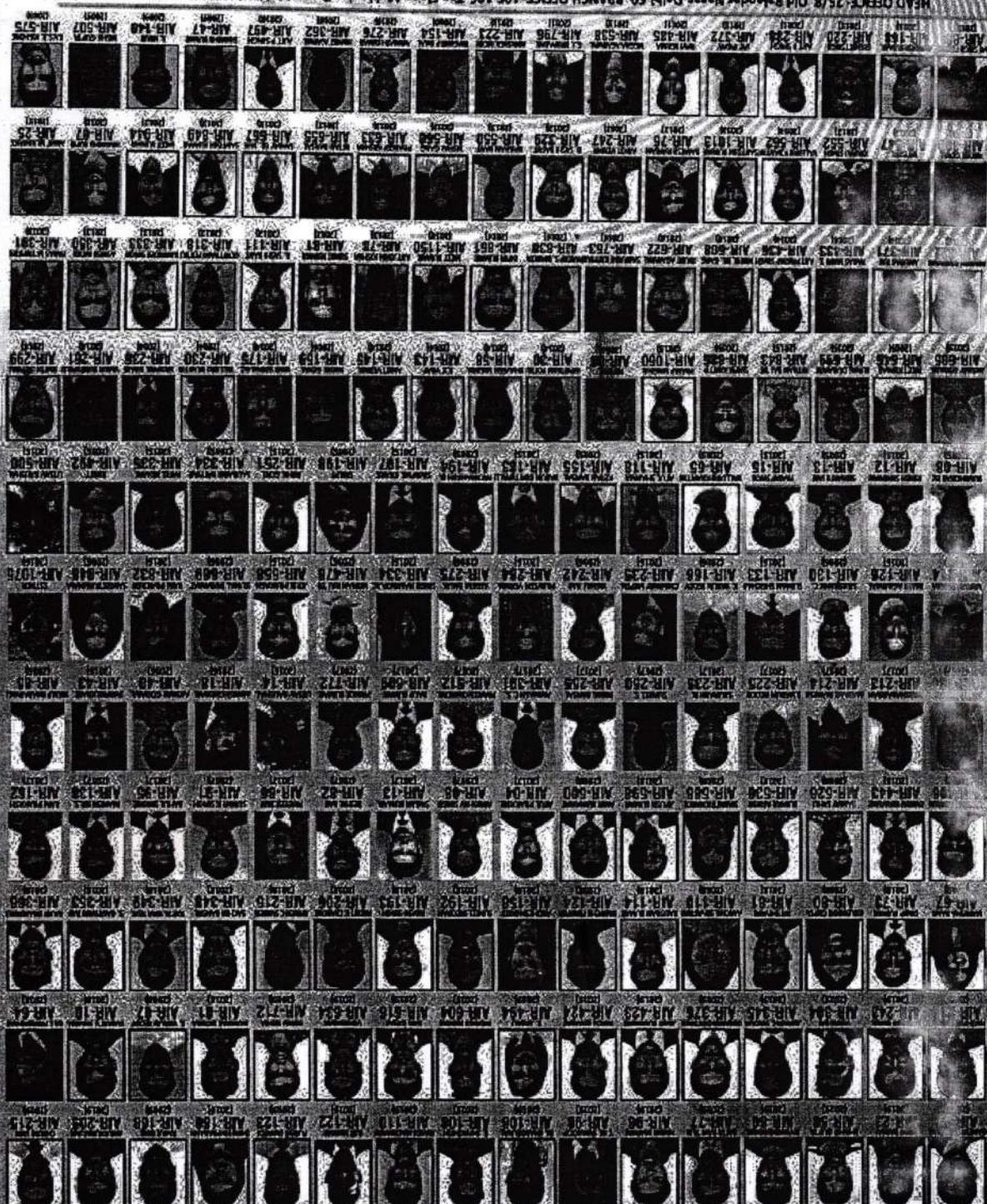
$$= 184 \pi \sin 90^\circ$$

$$= 184 \pi$$

(8) b
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