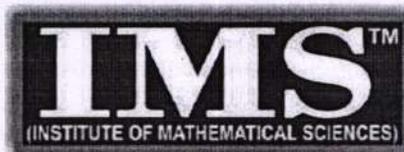


Date :

A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET



168
250

MAINS TEST SERIES-2022

ADVANCE TEST SERIES (MARCH.-2022 to MAY.-2022)

IAS/IFoS

MATHEMATICS

Under the guidance of K. Venkanna

LINEAR ALGEBRA, CALCULUS AND THREE DIMENSIONAL GEOMETRY

TEST CODE: TEST-1: IAS(M)/06-MARCH-2022

Time: 3 Hours

Maximum Marks: 250

INSTRUCTIONS

1. This question paper-cum-answer booklet has 58 pages and has **33 PART/SUBPART** questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated."
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any **THREE** of the remaining questions selecting at least **ONE** question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any page from this booklet.

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name Jankesh Jymera

Roll No.

Test Centre

Medium English

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the invigilator

IMPORTANT NOTE:

Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

P.T.O.

INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED
1	(a)			
	(b)			
	(c)			
	(d)			
	(e)			
2	(a)			
	(b)			
	(c)			
	(d)			
3	(a)			
	(b)			
	(c)			
	(d)			
4	(a)			
	(b)			
	(c)			
	(d)			
5	(a)			
	(b)			
	(c)			
	(d)			
	(e)			
6	(a)			
	(b)			
	(c)			
	(d)			
7	(a)			
	(b)			
	(c)			
	(d)			
8	(a)			
	(b)			
	(c)			
	(d)			
Total Marks				

SECTION - A

1. (a) Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigen values of a $n \times n$ square matrix A with corresponding eigen vectors X_1, X_2, \dots, X_n . If B is a matrix similar to A show that the eigen values of B are same as that of A . Also find the relation between the eigen vectors of B and eigen vectors of A . [10]

Given B is similar to A , then $B = CA\bar{C}^{-1}$ for some invertible C — (1)

Eigen values:

$$|B - \lambda I| = 0 \text{ where } \lambda \text{ are the eigen values of } B$$

$$\Rightarrow |CA\bar{C}^{-1} - \lambda I| = 0$$

$$\Rightarrow |CA\bar{C}^{-1} - C(\lambda I)\bar{C}^{-1}| = 0$$

$$\Rightarrow |C| |A - \lambda I| |\bar{C}^{-1}| = 0$$

$$\Rightarrow |A - \lambda I| = 0 \quad \left[\because |C| \neq 0 \text{ and } |\bar{C}^{-1}| = \frac{1}{|C|} \neq 0 \right]$$

\Rightarrow Eigen values of B are same as that of A , i.e., $\lambda_1, \lambda_2, \dots, \lambda_n$

Eigen vectors

$$B = CA\bar{C}^{-1}$$

$$\Rightarrow \bar{C}^{-1}B = A\bar{C}^{-1}$$

Let x be the eigen vector of A corresponding to λ
and y be the eigen vector of B corresponding to λ

$$A\bar{C}^{-1}(Cx) = A(x) = \lambda x$$

$$\Rightarrow (\bar{C}^{-1}B)(Cx) = \lambda x$$

$$B(Cx) = \lambda(Cx)$$

$\Rightarrow Cx = y \quad \because y$ is the eigen vector of B corresponding to λ

- 1/ (b) Are the vectors
 $\alpha_1 = (1, 1, 2, 4)$, $\alpha_2 = (2, -1, -5, 2)$
 $\alpha_3 = (1, -1, -4, 0)$, $\alpha_4 = (2, 1, 1, 6)$ linearly independent in \mathbb{R}^4 ?
 Find a basis for the subspace of \mathbb{R}^4 spanned by the four vectors.

[10]

Writing α_i 's in matrix form:

$$\begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & -5 & 2 \\ 1 & -1 & -4 & 0 \\ 2 & 1 & 1 & 6 \end{bmatrix}$$

Row Operations:

i) $R_2 \rightarrow R_2 - 2R_1$
 ii) $R_3 \rightarrow R_3 - R_1$
 iii) $R_4 \rightarrow R_4 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 0 & -2 & -6 & -4 \\ 0 & -1 & -3 & -2 \end{bmatrix}$$

i) $R_4 \rightarrow 3R_4 - R_2$
 ii) $R_3 \rightarrow R_3 - \frac{2}{3}R_2$
 iii) $R_2 \rightarrow R_2 / (-3)$

$$\begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The dimension of the space spanned by the given vectors is 2

They are linearly dependent.

Basis vectors: $\{ [1 \ 1 \ 2 \ 4], [0 \ 1 \ 3 \ 2] \}$

1. (c) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be such that

$$f(x) = \begin{cases} \frac{\sin(a+1)x + \sin x}{x} & , \text{if } x < 0 \\ c & , \text{if } x = 0 \\ \frac{(x+bx^2)^{1/2} - x^{1/2}}{bx^{1/2}} & , \text{if } x > 0 \end{cases}$$

Determine the values of a, b, c for which the function is continuous at $x = 0$.

[10]

Since the function is continuous at $x=0$;

$$\lim_{x \rightarrow 0^-} f(x) = c \Rightarrow \lim_{x \rightarrow 0^-} \frac{\sin(a+1)x + \sin x}{x} = c$$

$$\text{By L'Hopital Rule, } \lim_{x \rightarrow 0} \frac{(a+1)\cos(a+1)x + \cos x}{1} = c$$

$$\Rightarrow a+2 = c \quad \text{--- (1)}$$

$$\lim_{x \rightarrow 0^+} f(x) = c \Rightarrow \lim_{x \rightarrow 0^+} \frac{(x+bx^2)^{1/2} - x^{1/2}}{bx^{1/2}} = c$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{(1+bx)^{1/2} - 1}{bx} = c$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{(1+bx) - 1}{(bx) [(1+bx)^{1/2} + 1]} = c$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{bx}{(bx) [(1+bx)^{1/2} + 1]} = c$$

$$(b \neq 0) \Rightarrow \frac{1}{2} = c \quad \text{--- (2)}$$

$$\text{from (1) \& (2); } \underline{c = \frac{1}{2}, b \neq 0, a = -\frac{3}{2}}$$

1. (d) Prove that the function $f(x, y) = \sqrt{|xy|}$ is not differentiable at the point $(0, 0)$, but that f_x and f_y both exist at the origin and have the value 0. Hence deduce that these two partial derivatives are continuous except at the origin. [10]

$$f_x(0,0) = \lim_{h \rightarrow 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{|h \cdot 0|} - 0}{h} = \frac{0}{h} = 0$$

$$f_y(0,0) = \lim_{k \rightarrow 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k \rightarrow 0} \frac{\sqrt{|0 \cdot k|} - 0}{k} = \frac{0}{k} = 0$$

f_x and f_y exist at $(0,0)$ and are equal to zero.

If $f(x,y)$ is differentiable at $(0,0)$

$$f(h,k) - f(0,0) = \sqrt{|hk|} - 0 = hf_x + kf_y + \phi(h,k) ; h,k \rightarrow 0 \text{ \& } \phi(h,k) \rightarrow 0$$

Let $(h,k) = r(\cos\theta, \sin\theta)$; then $h,k \rightarrow 0$ as $r \rightarrow 0$

$$\Rightarrow f(r,\theta) - f(0,0) = \sqrt{|r^2 \cos\theta \sin\theta|} = (r \cos\theta)(0) + (r \sin\theta)(0) + \phi(r,\theta) \rightarrow 0 \text{ as } r \rightarrow 0$$

$$\Rightarrow r \sqrt{|\sin\theta \cos\theta|} = 0$$

$\Rightarrow \sin\theta \cos\theta = 0$ for arbitrary θ , which is not possible

$\therefore f(x,y)$ cannot be differentiable at $(0,0)$

$$f_x(x,y) = \lim_{h \rightarrow 0} \frac{f(x+h,y) - f(x,y)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{|(x+h)y|} - \sqrt{|xy|}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(x+h)y - xy}{h(\sqrt{|x+h|y} + \sqrt{|xy|})}$$

$$= \lim_{h \rightarrow 0} \frac{y}{\sqrt{|x+h|y} + \sqrt{|xy|}}$$

$$= \frac{y}{2\sqrt{|xy|}} = \frac{1}{2} \sqrt{\frac{|y|}{x}} \quad \text{--- (1)}$$

By symmetry, $f_y(x, y) = \frac{x}{2\sqrt{xy}} = \frac{1}{2}\sqrt{\frac{x}{y}} \quad \text{--- ②}$

from ①; $f_x(0, 0)$ is not defined

from ②; $f_y(0, 0)$ is not defined

But $f_x(0, 0) = f_y(0, 0) = 0$

$\therefore f_x$ and f_y are discontinuous at $(x, y) = (0, 0)$

1. (e) Prove that the centres of the spheres which touch the lines $y = mx, z = c$; $y = -mx, z = -c$ lie upon the conicoid $mxy + cz(1 + m^2) = 0$.

[10]

$y = mx, z = c$ & $y = -mx, z = -c$ touch the sphere

Let $O(a, b, c)$ be centre of the sphere,

then P_1 be the point of contact with $y = mx$

P_2 be " " " $z = c$

P_3 be " " " $z = -c$

the OP_1 is perpendicular to $y - mx = 0$ & $z = \pm c$

$$\Rightarrow (a, b, c) \parallel (-m, 1, 0)$$

$$(a, b, c) \parallel (0, 0, 1)$$

$$\Rightarrow \frac{a}{-m} = \frac{b}{1} = \frac{c}{0} = k_1$$

$$\& \frac{a}{0} = \frac{b}{0} = \frac{c}{1} = k_2$$



3. (a) (i) Find the diagonal form D and the diagonalizing matrix P for the following matrix over \mathbb{C} :

$$A = \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$$

- (ii) Let $U = \text{span} \{(1, 3, -2, 2, 3), (1, 4, -3, 4, 2), (2, 3, -1, -2, 9)\}$
 $W = \text{span} \{(1, 3, 0, 2, 1), (1, 5, -6, 6, 3), (2, 5, 3, 2, 1)\}$
 be the subspace of \mathbb{R}^5 .
 Find the basis and dimension of U , W , $U + W$ and $U \cap W$.

[17]

i) Let $A = PDP^{-1}$

$$|A - \lambda I| = 0 \text{ for eigen values}$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 4 \\ -4 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)^2 + 16 = 0$$

$$\Rightarrow \lambda^2 - 6\lambda + 25 = 0$$

$$\lambda = \frac{6 \pm \sqrt{36 - 4(25)}}{2}$$

$$\lambda = 3 \pm 4i$$

$$\boxed{\lambda_1 = 3 + 4i}$$

$(A - \lambda_1)X_1 = 0$ where $X_1 = \begin{bmatrix} x \\ y \end{bmatrix}^T$ is the eigen vector corresponding to λ_1 ; $x, y \in \mathbb{C}$

$$\begin{bmatrix} 3 - (3 + 4i) & 4 \\ -4 & 3 - (3 + 4i) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\begin{bmatrix} -4i & 4 \\ -4 & -4i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\Rightarrow -4ix + 4y = 0 \Rightarrow x + iy = 0$$

$$\text{Let } y = 1 \Rightarrow x = -i \Rightarrow X_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix} \text{ --- (1)}$$

$$\boxed{\lambda_2 = 3 - 4i}$$

$$(A - \lambda_2)X_2 = 0 \Rightarrow \begin{bmatrix} 3 - (3 - 4i) & 4 \\ -4 & 3 - (3 - 4i) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 4i & 4 \\ -4 & 4i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow x + iy = 0$$

$$\text{Let } x = 1 \Rightarrow y = -i \Rightarrow X_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix} \text{ --- (2)}$$

from ① & ② : $P = \begin{bmatrix} x_1 & x_2 \end{bmatrix} = \begin{bmatrix} -i & 1 \\ 1 & -i \end{bmatrix}$

$$O = \begin{bmatrix} 3+4i & 0 \\ 0 & 3-4i \end{bmatrix}$$

ii) $U = \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 1 & 4 & -3 & 4 & 2 \\ 2 & 3 & -1 & -2 & 9 \end{bmatrix}$

Row Operations:

$$\begin{array}{l} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - 2R_1 \end{array} \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & -3 & 3 & -6 & 3 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + 3R_2 \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row Echelon Form

$$\dim U = 2$$

Basis of $U = \left\{ \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \end{bmatrix} \right\}$

$$U+W = \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 1 & 3 & 0 & 2 & 1 \\ 0 & 1 & -3 & 2 & 1 \end{bmatrix}$$

Row Operations:

$$R_3 \rightarrow R_3 - R_1 \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 2 & 0 & -2 \\ 0 & 1 & -3 & 2 & 1 \end{bmatrix}$$

$$W = \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 1 & 5 & -6 & 6 & 3 \\ 2 & 5 & 3 & 2 & 1 \end{bmatrix}$$

Row Operations:

$$\begin{array}{l} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - 2R_1 \end{array} \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 2 & -6 & 4 & 2 \\ 0 & -1 & 3 & -2 & -1 \end{bmatrix}$$

$$\begin{array}{l} R_2 \rightarrow R_2 / 2 \\ R_3 \rightarrow 2R_3 + R_2 \end{array} \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 1 & -3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row Echelon Form

$$\dim W = 2$$

Basis of $W = \left\{ \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 1 & -3 & 2 & 1 \end{bmatrix} \right\}$

$$R_4 \rightarrow R_4 - R_2$$

$$R_3 \rightarrow R_3/2$$

$$\begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & -2 & 0 & 2 \end{bmatrix}$$

$$R_4 \rightarrow R_4 + 2R_3$$

$$\begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row Echelon form

$$\dim(U+W) = 3$$

$$\text{Basis of } U+W = \left\{ \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix} \right\}$$

$$\dim(U \cap W) = \dim(U) + \dim(W) - \dim(U+W)$$

$$= 2 + 2 - 3 = 1$$

$\Rightarrow \dim(U \cap W)$ has one basis vector

$$\text{Let } v \in U \cap W; v = (a, b, c, d, e)$$

$$v = x_1 (1 \ 3 \ -2 \ 2 \ 3) + y_1 (0 \ 1 \ 2 \ -1 \ -1)$$

$$= (x_1, 3x_1 + y_1, -2x_1 + 2y_1, 2x_1 - y_1, 3x_1 - y_1) \quad \text{--- (1)}$$

$$v = x_2 (1 \ 3 \ 0 \ 2 \ 1) + y_2 (0 \ 1 \ -3 \ 2 \ 1)$$

$$= (x_2, 3x_2 + y_2, -3y_2, 2x_2 + 2y_2, x_2 + y_2) \quad \text{--- (2)}$$

$$\text{from (1) & (2); } x_1 = x_2$$

$$3x_1 + y_1 = 3x_2 + y_2 \Rightarrow y_1 = y_2$$

$$-2x_1 + 2y_1 = -3y_2 \Rightarrow -2x_1 + 2y_1 = -3y_1 \Rightarrow -2x_1 = -5y_1 \Rightarrow x_1 = \frac{5}{2}y_1 \quad \text{--- (3)}$$

$$\text{Substituting (3) in (1) } \Rightarrow v = \left(\frac{5}{2}, \frac{17}{2}, -3, 4, \frac{13}{2} \right) y_1 = (5, 17, -6, 8, 13) y_1$$

$$\therefore \text{Basis of } U \cap W = \left\{ [5 \ 17 \ -6 \ 8 \ 13] \right\}$$

3. (b) (i) Show that if $a > 1$, $\int_0^{\infty} \frac{x^a}{a^x} dx = \frac{\Gamma(a+1)}{(\log a)^{a+1}}$.

(ii) If $v = At^{-1/2} e^{-x^2/4u^2t}$, prove that $\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}$ [16]

$$i) \int_0^{\infty} \frac{a^x}{a^x} dx$$

$$\text{let } a^x = y \Rightarrow x \log a = \log y$$

$$dx \log a = \frac{1}{y} dy$$

$$\Rightarrow \int_1^{\infty} \left[\frac{1}{y} \log a \right] \left[\frac{1}{y} \right] \left[\frac{\log y}{\log a} \right]^a dy$$

$$= \int_1^{\infty} \frac{1}{y^2} \frac{\log y}{(\log a)^{a+1}} dy$$

$$= \frac{1}{(\log a)^{a+1}} \int_1^{\infty} \frac{[\log y]^a}{y^2} dy$$

$$\text{let } \log y = x \Rightarrow y = e^x \Rightarrow dy = e^x dx$$

$$= \frac{1}{(\log a)^{a+1}} \int_0^{\infty} \frac{x^a}{e^{2x}} \cdot e^x dx$$

$$= \frac{1}{(\log a)^{a+1}} \int_0^{\infty} x^a e^{-x} dx$$

$$= \frac{1}{(\log a)^{a+1}} \Gamma(a+1)$$

$$ii) \text{ Given } v = A t^{-1/2} e^{-x^2/4at}$$

$$\begin{aligned} \frac{\partial v}{\partial t} &= -\frac{A}{2} t^{-3/2} e^{-x^2/4at} + A t^{-1/2} e^{-x^2/4at} \left[-\frac{x^2}{4a^2} \times -\frac{1}{t^2} \right] \\ &= A t^{-1/2} e^{-x^2/4at} \left[-\frac{1}{2t} + \frac{x^2}{4a^2 t^2} \right] \quad \text{--- (1)} \end{aligned}$$

$$\begin{aligned} \frac{\partial v}{\partial x} &= A t^{-1/2} e^{-x^2/4at} (-2x/4at) \\ &= -\frac{x A t^{-1/2}}{2a^2 t} e^{-x^2/4at} \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 v}{\partial x^2} &= -\frac{A t^{-1/2}}{2a^2 t} e^{-x^2/4at} - \frac{x A t^{-1/2}}{2a^2 t} e^{-x^2/4at} \left(-\frac{2x}{4a^2 t} \right) \\ &= -\frac{A t^{-3/2}}{2a^2} e^{-x^2/4at} + \frac{x^2}{4a^2} t^{-1/2} e^{-x^2/4at} \\ &= \frac{A t^{-1/2}}{a^2} e^{-x^2/4at} \left[-\frac{1}{2t} + \frac{x^2}{4a^2 t^2} \right] \quad \text{--- (2)} \end{aligned}$$

$$\text{from (1) \& (2): } \frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}$$

SECTION - B

5. (a) ✓ Verify the Cayley-Hamilton theorem for the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix}$$

Using this, show that A is non-singular and find A^{-1} .

[10]

Cayley - Hamilton Theorem: Every square matrix satisfies its characteristic equation

Characteristic equation: $|A - \lambda I| = 0$

$$\rightarrow \begin{vmatrix} 1-\lambda & 0 & -1 \\ 2 & 1-\lambda & 0 \\ 3 & -5 & 1-\lambda \end{vmatrix} = 0$$

$$\rightarrow (1-\lambda) [(1-\lambda)^2] - 1 [-10 - 3(1-\lambda)] = 0$$

$$\rightarrow (1-\lambda)^3 + [10 + 3 - 3\lambda] = 0$$

$$\rightarrow -\lambda^3 + 3\lambda^2 - 3\lambda + 1 + 13 - 3\lambda = 0$$

$$\rightarrow \lambda^3 - 3\lambda^2 + 6\lambda - 14 = 0$$

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 5 & -2 \\ 4 & 1 & -2 \\ -4 & -10 & -2 \end{bmatrix}$$

$$A^3 = A \cdot A^2 = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix} \begin{bmatrix} -2 & 5 & -2 \\ 4 & 1 & -2 \\ -4 & -10 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 15 & 0 \\ 0 & 11 & -6 \\ -30 & 0 & 2 \end{bmatrix}$$

$$A^3 - 3A^2 + 6A - 14I = \begin{bmatrix} 2+6+6-14 & 15-15+0-0 & 0+6-6-0 \\ 0-12+12-0 & 11-3+6-14 & -6+6+0+0 \\ -30+12+18-0 & 0+30-30-0 & 2+6+6-14 \end{bmatrix} = 0$$

\therefore Cayley Hamilton Theorem is Verified

$$A^3 - 3A^2 + 6A = 14I$$

$$\frac{1}{14} A^3 - 3A^2 + 6A = A^{-1}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} -2 & 5 & -2 \\ 4 & 1 & -2 \\ -4 & -10 & -2 \end{bmatrix} - \begin{bmatrix} 3 & 0 & -3 \\ 6 & 3 & 0 \\ 9 & -15 & 3 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \frac{1}{14} \begin{bmatrix} 1 & 5 & 1 \\ -2 & 4 & -2 \\ -13 & 5 & 1 \end{bmatrix}$$

5. (b) If $T : V_4 \rightarrow V_4$ defined by

$$T(x_1, x_2, x_3, x_4) = (0, 2x_1, 3x_1 + 2x_2, x_2 + 4x_3)$$

then prove that T is nilpotent of degree 4. Prove also that $I - T$ and $I + T$ are non-singular. [10]

$$T(x_1, x_2, x_3, x_4) = (0, 2x_1, 3x_1 + 2x_2, x_2 + 4x_3)$$

In matrix form $TX = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 1 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

$$T^2 = T \cdot T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 14 & 8 & 0 & 0 \end{bmatrix}$$

$$T^4 = T^2 \cdot T^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore T$ is nilpotent of degree 4.

$$I - T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -3 & -2 & 1 & 0 \\ 0 & -1 & -4 & 1 \end{bmatrix}$$

$$\det(I - T) = 1 \Rightarrow \underline{I - T \text{ is non-singular}}$$

(lower triangular matrix)

$$I + T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 0 & 1 & 4 & 1 \end{bmatrix}$$

$$\det(I + T) = 1 \Rightarrow \underline{I + T \text{ is non-singular}}$$

5. (e) Integrate the function $f(x,y) = xy(x^2 + y^2)$ over the domain $R : \{-3 \leq x^2 - y^2 \leq 3, 1 \leq xy \leq 4\}$.

[10]

$$\text{let } u = x^2 - y^2, v = xy$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix} = 2(x^2 + y^2)$$

$$\Rightarrow dudv = 2(x^2 + y^2) dx dy$$

$$\int_R xy(x^2 + y^2) dx dy = \int_1^4 \int_{-3}^3 (uv) \frac{dudv}{2}$$

$$= \int_1^4 \frac{v dv}{2} \int_{-3}^3 du$$

$$= \left[\frac{v^2}{4} \right]_1^4 \left[u \right]_{-3}^3$$

$$\Rightarrow \frac{15}{4} \times 6$$

$$\Rightarrow \frac{45}{2}$$

5. (d) A line makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube; prove that $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + \cos^2 \delta = 4/3$. [10]

Let this be a unit cube OABCDEFG

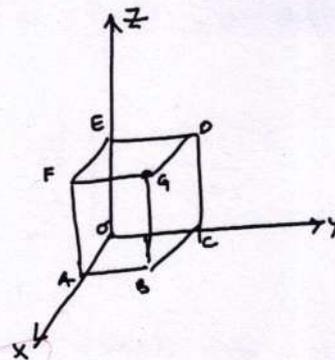
Direction ratios of diagonals:

$$OG = (1, 1, 1) - (0, 0, 0) = (1, 1, 1)$$

$$AO = (0, 1, 1) - (1, 0, 0) = (-1, 1, 1)$$

$$CF = (1, -1, 1)$$

$$BE = (1, 1, -1)$$



Let l, m, n be the Direction Cosines of the line,

$$\text{then } \cos \alpha = \frac{l+m+n}{\sqrt{3}} \quad (\because |OG| = |AO| = |CF| = |BE| = \sqrt{3})$$

$$\cos \beta = \frac{-l+m+n}{\sqrt{3}}$$

$$\cos \gamma = \frac{l-m+n}{\sqrt{3}}$$

$$\cos \delta = \frac{l+m-n}{\sqrt{3}}$$

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + \cos^2 \delta = \frac{1}{3} \left[(l+m+n)^2 + (-l+m+n)^2 + (l-m+n)^2 + (l+m-n)^2 \right]$$

$$= \frac{1}{3} \left[4(l^2+m^2+n^2) + 2(lm+mn+ln + lm-mn-ln + mn-lm-ln + ln-ml-mn) \right]$$

$$= \frac{4}{3} [l^2+m^2+n^2]$$

$$= \frac{4}{3}$$

5. (c) A sphere of constant radius r passes through the origin O and cuts the axes in A, B, C . Find the locus of the foot of the perpendicular from O to the plane ABC . [10]

Let the coordinates of A, B, C be $(a, 0, 0), (0, b, 0), (0, 0, c)$ respectively.

Let the equation of sphere be: $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

1) Since it passes through $(0, 0, 0) \Rightarrow d = 0$

2) It passes through $(a, 0, 0) \Rightarrow a^2 + 2ua = 0 \Rightarrow u = -a/2$

Similarly, $v = -b/2$

$w = -c/2$

\Rightarrow Equation of sphere is $x^2 + y^2 + z^2 - ax - by - cz = 0 \Rightarrow r^2 = a^2 + b^2 + c^2$ — (1)

Equation of plane ABC : $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \Rightarrow bcx + acy + abz = abc$

Let $P(x, y, z)$ be foot of perpendicular from O to ABC

then $OP \perp ABC$

$\Rightarrow (x, y, z) \parallel$ normal of plane

$$\Rightarrow \frac{x}{1/a} = \frac{y}{1/b} = \frac{z}{1/c} = k$$

$$\Rightarrow ax = by = cz = k$$

$$\Rightarrow a = \frac{k}{x}, b = \frac{k}{y}, c = \frac{k}{z}$$

$$\text{Since } a^2 + b^2 + c^2 = r^2 \Rightarrow k^2 \left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} \right) = r^2$$

$$\Rightarrow \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \text{constant}$$

6. (a) Find a Hermitian and a skew-Hermitian matrix each whose sum is the matrix

$$\begin{bmatrix} 2i & 3 & -1 \\ 1 & 2+3i & 2 \\ -i+1 & 4 & 5i \end{bmatrix}$$

[08]

Let the given matrix be A

Let P, Q be Hermitian & skew Hermitian matrices respectively,

then $P^{\theta} = P, Q^{\theta} = -Q$

$$P + Q = A \quad \text{--- (1)}$$

$$(P + Q)^{\theta} = A^{\theta}$$

$$\rightarrow P^{\theta} + Q^{\theta} = A^{\theta}$$

$$\rightarrow P - Q = A^{\theta} \quad \text{--- (2)}$$

from (1) & (2) $P = \frac{A + A^{\theta}}{2}, Q = \frac{A - A^{\theta}}{2}$

$$A = \begin{bmatrix} 2i & 3 & -1 \\ 1 & 2+3i & 2 \\ -i+1 & 4 & 5i \end{bmatrix}; A^{\theta} = \begin{bmatrix} -2i & 1 & i+1 \\ 3 & 2-3i & 4 \\ -1 & 2 & -5i \end{bmatrix}$$

$$P = \frac{A + A^{\theta}}{2} = \frac{1}{2} \begin{bmatrix} 0 & 4 & i \\ 4 & 4 & 6 \\ -i & 6 & 0 \end{bmatrix}$$

$$Q = \frac{A - A^{\theta}}{2} = \frac{1}{2} \begin{bmatrix} 4i & 2 & -i-2 \\ -2 & 6i & -2 \\ -i+2 & 2 & 10i \end{bmatrix}$$

6. (b) Let V be the vector space of all 2×2 matrices-over the field F . Let W_1 be the set of matrices of the form $\begin{bmatrix} x & -x \\ y & z \end{bmatrix}$ and let W_2 be the set of matrices of the form

$$\begin{bmatrix} a & b \\ -a & c \end{bmatrix}.$$

- (i) Prove that W_1 and W_2 are subspaces of V .
 (ii) Find the dimensions of W_1 , W_2 , $W_1 + W_2$, and $W_1 \cap W_2$. [12]

i) $W_1 \subseteq V$ & $W_2 \subseteq V$

$$\text{Let } w_{11} = \begin{bmatrix} x_1 & -x_1 \\ y_1 & z_1 \end{bmatrix}, w_{12} = \begin{bmatrix} x_2 & -x_2 \\ y_2 & z_2 \end{bmatrix} \in W_1$$

$$\text{Then i) } -w_{12} = \begin{bmatrix} -x_2 & x_2 \\ y_2 & z_2 \end{bmatrix} \in W_1$$

$$\text{ii) } w_{11} - w_{12} = \begin{bmatrix} x_1 - x_2 & -(x_1 - x_2) \\ y_1 - y_2 & z_1 - z_2 \end{bmatrix} \in W_1$$

$\therefore W_1$ is a subspace of V

$$\text{Let } w_{21} = \begin{bmatrix} a_1 & b_1 \\ -a_1 & c_1 \end{bmatrix}, w_{22} = \begin{bmatrix} a_2 & b_2 \\ -a_2 & c_2 \end{bmatrix} \in W_2$$

$$\text{i) } -w_{21} = \begin{bmatrix} -a_1 & -b_1 \\ a_1 & -c_1 \end{bmatrix} \in W_2$$

$$\text{ii) } w_{21} - w_{22} = \begin{bmatrix} a_1 - a_2 & b_1 - b_2 \\ -(a_1 - a_2) & c_1 - c_2 \end{bmatrix} \in W_2$$

$\therefore W_2$ is a subspace of V

ii) Let $w_1 = \begin{bmatrix} x & -x \\ y & z \end{bmatrix} \in W_1$

$$\text{Then } w_1 = x \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + y \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + z \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{Then } \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \text{ is the basis of } W_1 \Rightarrow \underline{\underline{\dim W_1 = 3}}$$

$$w_2 = \begin{bmatrix} a & b \\ -a & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{Then } \left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \text{ is the basis of } W_2 \Rightarrow \underline{\underline{\dim W_2 = 3}}$$

$$W_1 \cap W_2 = \left\{ \begin{bmatrix} x & -x \\ -x & y \end{bmatrix} \right\} = x \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} + y \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \underline{\underline{\dim W_1 \cap W_2 = 2}}$$

$$W_1 + W_2 = \left\{ \begin{bmatrix} a & -z \\ y & z \end{bmatrix} + \begin{bmatrix} a & b \\ -a & c \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} a+a & b-z \\ y-a & z+c \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} p & q \\ r & s \end{bmatrix} \right\} \text{ where } p=a+a, q=b-z, r=y-a, s=z+c \in F$$

$W_1 + W_2$ represent the vector space of all 2×2 matrices $\Rightarrow \underline{\underline{\dim W_1 + W_2 = 4}}$

6. (c) Consider the singular matrix

$$A = \begin{bmatrix} -1 & 3 & -1 & 1 \\ -3 & 5 & 1 & -1 \\ 10 & -10 & -10 & 14 \\ 4 & -4 & -4 & 8 \end{bmatrix}$$

Given that one eigenvalue of A is 4 and one eigenvector that does not correspond to this eigenvalue 4 is $(1 \ 1 \ 0 \ 0)^T$. Find all the eigenvalues of A other than 4 and hence also find the real numbers p, q, r that satisfy the matrix equation $A^4 + pA^3 + qA^2 + rA = 0$ [15]

Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be \neq eigen values

$$\rightarrow \lambda_1 = 4 \text{ --- (1)}$$

$$\rightarrow \text{Let } Ax_2 = \lambda_2 x_2 \Rightarrow \begin{bmatrix} -1 & 3 & -1 & 1 \\ -3 & 5 & 1 & -1 \\ 10 & -10 & -10 & 14 \\ 4 & -4 & -4 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \lambda_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \lambda_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \lambda_2 = 2 \text{ --- (2)}$$

\rightarrow We know sum of eigen values = Trace of A

$$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = -1 + 5 - 10 + 8 \Rightarrow 6 + \lambda_3 + \lambda_4 = 2 \quad (\text{from (1) \& (2)})$$

$$\Rightarrow \lambda_3 + \lambda_4 = -4 \text{ --- (3)}$$

\rightarrow We know product of eigen values is equal to determinant of the matrix

$$\begin{aligned} \det A &= \begin{vmatrix} -1 & 3 & -1 & 1 \\ -3 & 5 & 1 & -1 \\ 10 & -10 & -10 & 14 \\ 0 & 0 & 0 & \frac{12}{5} \end{vmatrix} = \frac{12}{5} [10(8) + 10(-1-3) - 10(-5+1)] \\ &= \frac{12}{5} [10] [8-4-4] = 0 \end{aligned}$$

$$\Rightarrow \lambda_1 \lambda_2 \lambda_3 \lambda_4 = 0$$

$$\Rightarrow \lambda_3 \lambda_4 = 0 \quad (\because \lambda_1, \lambda_2 \neq 0)$$

$$\text{Let } \lambda_4 = 0 \text{ --- (4)}$$

$$\text{from (3) \& (4); } \lambda_3 = -4 \text{ --- (5)}$$

By Cayley-Hamilton theorem, Every square matrix satisfies its characteristic equation:

$$\Rightarrow A^4 - (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)A^3 + (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_4 + \lambda_1\lambda_4)A^2 - (\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4)A + (\lambda_1\lambda_2\lambda_3\lambda_4)I = 0$$

$$\Rightarrow A^4 - (2)A^3 + (8 + (-8) + 0 + 0)A^2 - (0 + 0 + 0 + (-32))A + 0(I) = 0$$

$$\Rightarrow A^4 - 2A^3 + 0A^2 + 32A = 0$$

$$\therefore \begin{aligned} p &= -2 \\ q &= 0 \\ r &= 32 \end{aligned}$$

6. (d) Consider the vector space $V(\mathbb{R})$ of all 2×2 matrices over the field \mathbb{R} of real numbers. Let T be a linear transformation on V that sends each matrix X onto AX , where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Find the matrix of T relative to the ordered basis B , where

$$B = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

[15]

Given $T(x) = AX$

$$T \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \quad \text{where } x_i \in \mathbb{R}$$

$$= \begin{bmatrix} x_1 + x_3 & x_2 + x_4 \\ x_1 + x_3 & x_2 + x_4 \end{bmatrix}$$

$$= (x_1 + x_3) \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + (x_2 + x_4) \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

let $x_1 + x_3 = x$

$x_2 + x_4 = y$

$$\Rightarrow T = x(\alpha_1 + \alpha_3) + y(\alpha_2 + \alpha_4)$$

$$\Rightarrow T = x\alpha_1 + x\alpha_3 + y\alpha_2 + y\alpha_4 \quad \text{for } x, y \in \mathbb{R}$$

7. (a) Evaluate $\int_0^1 \tan^{-1}\left(1 - \frac{1}{x}\right) dx$

[13]

$$\begin{aligned}
 I &= \int_0^1 \tan^{-1}\left(1 - \frac{1}{x}\right) dx = \int_0^1 \tan^{-1}\left(\frac{x-1}{x}\right) dx \\
 &= \int_0^1 \tan^{-1}\left(\frac{(1-x)-1}{(1-x)}\right) dx \\
 &= \int_0^1 \tan^{-1}\left(\frac{x}{x-1}\right) dx
 \end{aligned}$$

$$2I = \int_0^1 \left[\tan^{-1}\left(\frac{x-1}{x}\right) + \tan^{-1}\left(\frac{x}{x-1}\right) \right] dx$$

$$= \int_0^1 \tan^{-1}\left[\frac{\frac{x-1}{x} + \frac{x}{x-1}}{1 - \left(\frac{x-1}{x}\right)\left(\frac{x}{x-1}\right)} \right] dx$$

$$= \int_0^1 \tan^{-1}\left(\frac{(x-1)^2 + x^2}{x(x-1) \cdot 0}\right) dx$$

$$= \int_0^1 \tan^{-1}(-\infty) dx \quad \left(\begin{array}{l} \because \text{numerator} > 0 \\ \text{denominator: } x(x-1) < 0 \end{array} \right)$$

$$= \int_0^1 -\frac{\pi}{2} dx$$

$$= -\frac{\pi}{2}$$

$$\Rightarrow I = \frac{-\pi}{4}$$

$$\begin{aligned}
 \frac{\sec\theta - 1}{(\sec\theta)(\sec\theta + 1)} &= \frac{x-1}{x} \\
 \frac{\tan\theta}{(\sec\theta)(\sec\theta + 1)} &= \frac{x}{x-1} \\
 x &= \sec^2\theta \\
 \sqrt{x} &= \sec\theta \\
 \frac{\tan^2\theta}{\sec^2\theta} &= \frac{x}{x-1} \\
 \tan^2(\sin^2\theta) &
 \end{aligned}$$

7. (b) Find the maximum and minimum values of $f(x, y, z) = x - 2y + 5z$ on the sphere $x^2 + y^2 + z^2 = 30$ by using Lagrange's multiplier method. [15]

$$F(x, y, z, \lambda) = x - 2y + 5z + \lambda (x^2 + y^2 + z^2 - 30) \quad \text{--- (1) (Lagrange Multiplier equation)}$$

$$\text{At extrema, } \frac{\partial F}{\partial x} = 0 \Rightarrow 1 + 2\lambda x = 0 \quad \text{--- (2)}$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow -2 + 2\lambda y = 0 \Rightarrow \lambda y - 1 = 0 \quad \text{--- (3)}$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow 5 + 2\lambda z = 0 \quad \text{--- (4)}$$

$$\frac{\partial F}{\partial \lambda} = 0 \Rightarrow x^2 + y^2 + z^2 = 30$$

$$\text{from (2), (3), (4) } \Rightarrow \left(\frac{-1}{2\lambda}\right)^2 + \left(\frac{1}{\lambda}\right)^2 + \left(\frac{-5}{2\lambda}\right)^2 = 30$$

$$\Rightarrow 1 + 4 + 25 = 30(4\lambda^2)$$

$$\Rightarrow 4\lambda^2 = 1 \Rightarrow \lambda = \pm \frac{1}{2} \text{ at extrema}$$

$$\lambda = \frac{1}{2} \Rightarrow x = -1, y = 2, z = -5 \Rightarrow (-1, 2, -5)$$

$$\lambda = -\frac{1}{2} \Rightarrow x = 1, y = -2, z = 5 \Rightarrow (1, -2, 5)$$

$$f(-1, 2, -5) = -1 - 2(2) + 5(-5) = -1 - 4 - 25 = -30 \text{ (minima)}$$

$$f(1, -2, 5) = 1 + 2(-2) + 5(5) = 1 - 4 + 25 = 22 \text{ (maxima)}$$

7. (c) Express $\int_0^1 x^m (1-x^n)^p dx$ in terms of Gamma function and hence evaluate the integral

$$\int_0^1 x^6 \sqrt{1-x^2} dx.$$

[10]

$$\text{Let } 1-x^n = y \Rightarrow -nx^{n-1} dx = dy$$

$$\Rightarrow x^n = 1-y$$

$$\Rightarrow x = (1-y)^{1/n}$$

$$\int_0^1 x^n (1-x^n)^p dx = \int_0^1 (1-y)^{m/n} y^p \frac{dy}{n(1-y)^{n-1}}$$

$$= \frac{1}{n} \int_0^1 (1-y)^{\frac{m-n+1}{n}} y^p dy$$

$$= \frac{1}{n} \int_0^1 y^p (1-y)^{\frac{m-n+1}{n}} dy$$

$$= \frac{B(p+1, \frac{m+1}{n})}{n}$$

$$= \frac{\Gamma(p+1) \Gamma(\frac{m+1}{n})}{n \Gamma(p+1 + \frac{m+1}{n})}$$

$$\int_0^1 x^6 (1-x^2)^{1/2} dx \Rightarrow \begin{matrix} m=6 \\ n=2 \\ p=1/2 \end{matrix} \Rightarrow \frac{\Gamma(\frac{3}{2}) \Gamma(\frac{7}{2})}{(2) \Gamma(\frac{3}{2} + \frac{7}{2})}$$

$$= \frac{\frac{1}{2} \Gamma(\frac{1}{2}) \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma(\frac{1}{2})}{2 \Gamma(5)}$$

$$= \frac{\frac{5}{16} (\sqrt{\pi})^2}{4 \times 3 \times 2 \times 1}$$

$$= \frac{5\pi}{108}$$

$$= \frac{5\pi}{108}$$

7. (d) Find the asymptotes of the curve :

$$y^2(x^2 - a^2) = x^2(x^2 - 4a^2)$$

[12]

$$\Rightarrow y^2 x^2 - y^2 a^2 = x^4 - 4a^2 x^2$$

$$\Rightarrow \underbrace{y^2 x^2 - x^4}_{\text{Fourth degree}} - \underbrace{y^2 a^2 + 4a^2 x^2}_{\text{Second degree}} = 0$$

By inspection, the asymptotes are given by $y^2 x^2 - x^4 = 0$

$$\Rightarrow x^2 (y^2 - x^2) = 0$$

$\Rightarrow y - x = 0, y + x = 0$ are the asymptotes, oblique to coordinate axes

Asymptotes parallel to x-axis:

Coefficient of highest degree x-term: 1 \Rightarrow no asymptotes parallel to x-axis

Asymptotes parallel to y-axis:

Coefficient of highest degree y-term: $(x^2 - a^2) = 0 \Rightarrow x - a = 0, x + a = 0$ are asymptotes parallel to y-axis

No. 1 INSTITUTE FOR IAS/IFoS EXAMINATIONS



OUR ACHIEVEMENTS IN IAS (FROM 2008 TO 2020)

 AIR-16 (2018)	 AIR-30 (2020)	 AIR-31 (2020)	 AIR-37 (2020)	 AIR-45 (2020)	 AIR-55 (2020)	 AIR-105 (2020)	 AIR-106 (2020)	 AIR-239 (2020)	 AIR-284 (2020)	 AIR-311 (2020)	 AIR-334 (2020)	 AIR-338 (2020)	 AIR-348 (2020)	 AIR-420 (2020)	 AIR-488 (2020)	 AIR-616 (2020)		
 AIR-07 (2021)	 AIR-23 (2021)	 AIR-50 (2021)	 AIR-69 (2021)	 AIR-77 (2021)	 AIR-96 (2021)	 AIR-98 (2021)	 AIR-106 (2021)	 AIR-108 (2021)	 AIR-110 (2021)	 AIR-122 (2021)	 AIR-123 (2021)	 AIR-166 (2021)	 AIR-168 (2021)	 AIR-205 (2021)	 AIR-215 (2021)	 AIR-216 (2021)	 AIR-243 (2021)	
 AIR-304 (2021)	 AIR-345 (2021)	 AIR-376 (2021)	 AIR-423 (2021)	 AIR-424 (2021)	 AIR-434 (2021)	 AIR-434 (2021)	 AIR-516 (2021)	 AIR-534 (2021)	 AIR-571 (2021)	 AIR-712 (2021)	 AIR-01 (2021)	 AIR-07 (2021)	 AIR-18 (2021)	 AIR-44 (2021)	 AIR-67 (2021)	 AIR-73 (2021)	 AIR-80 (2021)	 AIR-81 (2021)
 AIR-110 (2021)	 AIR-114 (2021)	 AIR-124 (2021)	 AIR-158 (2021)	 AIR-192 (2021)	 AIR-193 (2021)	 AIR-206 (2021)	 AIR-215 (2021)	 AIR-215 (2021)	 AIR-348 (2021)	 AIR-349 (2021)	 AIR-353 (2021)	 AIR-366 (2021)	 AIR-406 (2021)	 AIR-443 (2021)	 AIR-525 (2021)	 AIR-536 (2021)	 AIR-586 (2021)	 AIR-588 (2021)
 AIR-000 (2017)	 AIR-04 (2017)	 AIR-08 (2017)	 AIR-13 (2017)	 AIR-02 (2017)	 AIR-08 (2017)	 AIR-09 (2017)	 AIR-09 (2017)	 AIR-139 (2017)	 AIR-139 (2017)	 AIR-182 (2017)	 AIR-184 (2017)	 AIR-213 (2017)	 AIR-214 (2017)	 AIR-225 (2017)	 AIR-235 (2017)	 AIR-250 (2017)	 AIR-256 (2017)	 AIR-301 (2017)
 AIR-512 (2017)	 AIR-509 (2017)	 AIR-772 (2017)	 AIR-14 (2017)	 AIR-18 (2017)	 AIR-49 (2017)	 AIR-43 (2017)	 AIR-55 (2017)	 AIR-114 (2017)	 AIR-126 (2017)	 AIR-130 (2017)	 AIR-133 (2017)	 AIR-166 (2017)	 AIR-235 (2017)	 AIR-242 (2017)	 AIR-264 (2017)	 AIR-275 (2017)	 AIR-334 (2017)	
 AIR-478 (2017)	 AIR-558 (2017)	 AIR-609 (2017)	 AIR-632 (2017)	 AIR-646 (2017)	 AIR-1075 (2017)	 AIR-08 (2017)	 AIR-12 (2017)	 AIR-13 (2017)	 AIR-15 (2017)	 AIR-05 (2017)	 AIR-118 (2017)	 AIR-155 (2017)	 AIR-183 (2017)	 AIR-194 (2017)	 AIR-197 (2017)	 AIR-198 (2017)	 AIR-231 (2017)	
 AIR-334 (2015)	 AIR-335 (2015)	 AIR-492 (2015)	 AIR-500 (2015)	 AIR-605 (2015)	 AIR-646 (2015)	 AIR-599 (2015)	 AIR-843 (2015)	 AIR-836 (2015)	 AIR-1060 (2015)	 AIR-08 (2015)	 AIR-30 (2015)	 AIR-58 (2015)	 AIR-143 (2015)	 AIR-145 (2015)	 AIR-159 (2015)	 AIR-175 (2015)	 AIR-230 (2015)	
 AIR-238 (2014)	 AIR-261 (2014)	 AIR-299 (2014)	 AIR-322 (2014)	 AIR-371 (2014)	 AIR-433 (2014)	 AIR-436 (2014)	 AIR-606 (2014)	 AIR-622 (2014)	 AIR-763 (2014)	 AIR-830 (2014)	 AIR-861 (2014)	 AIR-1150 (2014)	 AIR-78 (2014)	 AIR-81 (2014)	 AIR-111 (2014)	 AIR-318 (2014)	 AIR-333 (2014)	
 AIR-350 (2013)	 AIR-391 (2013)	 AIR-398 (2013)	 AIR-547 (2013)	 AIR-552 (2013)	 AIR-582 (2013)	 AIR-1013 (2013)	 AIR-78 (2013)	 AIR-247 (2013)	 AIR-329 (2013)	 AIR-350 (2013)	 AIR-560 (2013)	 AIR-633 (2013)	 AIR-655 (2013)	 AIR-667 (2013)	 AIR-849 (2013)	 AIR-944 (2013)	 AIR-07 (2013)	
 AIR-25 (2011)	 AIR-88 (2011)	 AIR-188 (2011)	 AIR-229 (2011)	 AIR-269 (2011)	 AIR-372 (2011)	 AIR-425 (2011)	 AIR-538 (2011)	 AIR-796 (2011)	 AIR-223 (2011)	 AIR-154 (2011)	 AIR-278 (2011)	 AIR-362 (2011)	 AIR-497 (2011)	 AIR-47 (2011)	 AIR-148 (2011)	 AIR-507 (2011)	 AIR-575 (2011)	

HEAD OFFICE: 25/B, Old Rajender Nagar, Delhi-60. BRANCH OFFICE: 105-106, Top Floor, Mukherjee Tower Mukherjee Nagar, Delhi-9

Ph.: 011-45629987, 9999197625 www.ims4maths.com e-Mail: ims4maths@gmail.com

Regional Office: H.No. 1-30-237, 2nd Floor, Room No. 202 R.K.S. Rancham's Blue Sapphire Ashok Nagar, Hyderabad-20. Ph.: 9652351152, 9652661152