Indian Forest Service Examination-2012

D-GT-M-NUA

MATHEMATICS

Paper I

Time Allowed : Three Hours   Maximum Marks : 200

INSTRUCTIONS

Candidates should attempt Question Nos. 1 and 5 which are compulsory and any THREE of the remaining questions, selecting at least ONE question from each Section.

All questions carry equal marks.

Marks allotted to parts of a question are indicated against each.

Answers must be written in ENGLISH only.

Assume suitable data, if considered necessary, and indicate the same clearly.

Unless indicated otherwise, symbols & notations carry their usual meaning.

Important Note : All parts/sub-parts of a question must be attempted contiguously. That is, candidates must finish attempting all the parts/sub-parts of each question they are answering in the answer-book before moving on to the next question.

Pages left blank must be clearly struck out. Answers that follow any pages left blank may not be given credit.

(Contd.)

Section – A

1. (a) Let $V = \mathbb{R}^3$ and $\alpha_1 = (1, 1, 2)$, $\alpha_2 = (0, 1, 3)$, $\alpha_3 = (2, 4, 5)$ and $\alpha_4 = (-1, 0, -1)$ be the elements of V. Find a basis for the intersection of the subspace spanned by $\{\alpha_1, \alpha_2\}$ and $\{\alpha_3, \alpha_4\}$. ⠀⠀⠀⠀8

   (b) Show that the set of all functions which satisfy the differential equation,

   $$\frac{d^2 f}{dx^2} + 3\frac{df}{dx} = 0 \text{ is a vector space.} \qquad 8$$

   (c) If the three thermodynamic variables P, V, T are connected by a relation, $f(P, V, T) = 0$ show that, $\left(\dfrac{\partial P}{\partial T}\right)_V \left(\dfrac{\partial T}{\partial V}\right)_P \left(\dfrac{\partial V}{\partial P}\right)_T = -1.$ ⠀⠀8

   (d) If $u = A\bar{e}^{8x}\sin(nt - gx)$, where A, g, n are positive constants, satisfies the heat conduction equation, $\dfrac{\partial u}{\partial t} = \mu\dfrac{\partial^2 u}{\partial x^2}$ then show that

   $$g = \sqrt{\left(\frac{n}{2\mu}\right)}. \qquad 8$$

   (e) Find the equations to the lines in which the plane $2x + y - z = 0$ cuts the cone
   $$4x^2 - y^2 + 3z^2 = 0. \qquad 8$$

2. (a) Let $f: \mathbb{R} \to \mathbb{R}^3$ be a linear transformation defined by $f(a, b, c) = (a, a + b, 0)$.

Find the matrices A and B respectively of the linear transformation f with respect to the standard basis (e_1, e_2, e_3) and the basis (e_1', e_2', e_3') where $e_1' = (1, 1, 0)$, $e_2' = (0, 1, 1)$, $e_3' = (1, 1, 1)$.

Also, show that there exists an invertible matrix P such that

$$B = \overset{-1}{P}AP \qquad 10$$

(b) Verify Cayley–Hamilton theorem for the matrix

$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ and find its inverse. Also express

$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ as a linear polynomial in A. 10

(c) Find the equations of the tangent plane to the ellipsoid $2x^2 + 6y^2 + 3z^2 = 27$ which passes through the line $x - y - z = 0 = x - y + 2z - 9$. 10

(d) Show that there are three real values of λ for which the equations :
$(a - \lambda) x + by + cz = 0$, $bx + (c - \lambda) y + az = 0$, $cx + ay + (b - \lambda) z = 0$ are simultaneously true and that the product of these values of λ is

$$D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

10

3. (a) Find the matrix representation of linear transformation T on V_3 ($\mathbb{R}$) defined as
$T(a, b, c) = (2b + c, a - 4b, 3a)$
corresponding to the basis
$B = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$.   10

(b) Find the dimensions of the rectangular box, open at the top, of maximum capacity whose surface is 432 sq. cm.   10

(c) If $2C$ is the shortest distance between the lines

$\dfrac{x}{l} - \dfrac{z}{n} = 1,\ y = 0$

and $\dfrac{y}{m} + \dfrac{z}{n} = 1,\ x = 0$

then show that

$\dfrac{1}{l^2} + \dfrac{1}{m^2} + \dfrac{1}{n^2} = \dfrac{1}{c^2}$.   10

(d) Show that the function defined as

$f(x) = \begin{cases} \dfrac{\sin 2x}{x} & \text{when } x \neq 0 \\ 1 & \text{when } x = 0 \end{cases}$

has removable discontinuity at the origin.   10

4. (a) Find by triple Integration the volume cut off from the cylinder $x^2 + y^2 = ax$ by the planes $z = mx$ and $z = nx$.   10

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(b) Show that all the spheres, that can be drawn through the origin and each set of points where planes parallel to the plane $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 0$ cut the co-ordinate axes, form a system of spheres which are cut orthogonally by the sphere

$$x^2 + y^2 + 2fx + 2gy + 2hz = 0$$

if $af + bg + ch = 0$.   10

(c) A plane makes equal intercepts on the positive parts of the axes and touches the ellipsoid $x^2 + 4y^2 + 9z^2 = 36$. Find its equation.   10

(d) Evaluate the following in terms of Gamma function :

$$\int_0^a \sqrt{\left(\frac{x^3}{a^3 - x^3}\right)}\, dx.$$

     10

Section – B

5. (a) Solve $\dfrac{dy}{dx} - \dfrac{\tan y}{1+x} = (1+x)e^x \sec y$.   8

(b) Solve and find the singular solution of $x^3 p^2 + x^2 py + a^3 = 0$.   8

(c) A particle is projected vertically upwards from the earth's surface with a velocity just sufficient to carry it to infinity. Prove that the time it takes to reach a height h is

$$\frac{1}{3}\sqrt{\left(\frac{2a}{g}\right)}\left[\left(1 + \frac{h}{a}\right)^{3/2} - 1\right].$$

     8

(d) A triangle ABC is immersed in a liquid with the vertex C in the surface and the sides AC, BC equally inclined to the surface. Show that the vertical C divides the triangle into two others, the fluid pressures on which are as $b^3 + 3ab^2 : a^3 + 3a^2b$ where a and b are the sides BC & AC respectively. 8

(e) If $u = x + y + z$, $v = x^2 + y^2 + z^2$,
$w = yz + zx + xy$,
prove that grad u, grad v and grad w are coplanar. 8

6. (a) Solve :

$$x^2 y \frac{d^2 y}{dx^2} + \left(x \frac{dy}{dx} - y \right)^2 = 0.$$ 10

(b) Find the value of $\iint_s \left(\vec{\nabla} \times \vec{F} \right) \cdot \vec{ds}$

taken over the upper portion of the surface $x^2 + y^2 - 2ax + az = 0$ and the bounding curve lies in the plane $z = 0$, when

$$\vec{F} = \left(y^2 + z^2 - x \right) \vec{i} + \left(z^2 + x^2 - y^2 \right) \vec{j}$$

$$+ \left(x^2 + y^2 - z^2 \right) \vec{k}.$$ 10

(c) A particle is projected with a velocity u and strikes at right angle on a plane through the plane of projection inclined at an angle β to the horizon. Show that the time of flight is

$$\frac{2u}{g\sqrt{\left(1+3\sin^2\beta\right)}},$$

range on the plane is $\dfrac{2u^2}{g}\cdot\dfrac{\sin\beta}{1+3\sin^2\beta}$

and the vertical height of the point struck is

$$\frac{2u^2\sin^2\beta}{g\left(1+3\sin^2\beta\right)}$$ above the point of projection.

10

(d) Solve $\dfrac{d^4y}{dx^4}+2\dfrac{d^2y}{dx^2}+y=x^2\cos x$. 10

7. (a) A particle is moving with central acceleration $\mu\left[r^5-c^4r\right]$ being projected from an apse at a distance c with velocity $\sqrt{\left(\dfrac{2\mu}{3}\right)c^3}$, show that

its path is a curve, $x^4+y^4=c^4$. 13

(b) A thin equilateral rectangular plate of uniform thickness and density rests with one end of its base on a rough horizontal plane and the other against a small vertical wall. Show that the least angle, its base can make with the horizontal plane is given by
$$\cot\theta=2\mu+\frac{1}{\sqrt{3}}$$
μ, being the coefficient of friction. 14

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(c) A semicircular area of radius a is immersed vertically with its diameter horizontal at a depth b. If the circumference be below the centre, prove that the depth of centre of pressure is

$$\frac{1}{4}\frac{3\pi\left(a^2+4b^2\right)+32ab}{4a+3\pi b}.$$

13.

8. (a) Solve $x = y\dfrac{dy}{dx} - \left(\dfrac{dy}{dx}\right)^2$.   10

(b) Find the value of the line integral over a circular path given by $x^2 + y^2 = a^2$, $z = 0$, where the vector field,

$$\overrightarrow{F} = (\sin y)\,\overrightarrow{i} + x(1 + \cos y)\,\overrightarrow{j}.$$

10

(c) A heavy elastic string, whose natural length is $2\pi a$, is placed round a smooth cone whose axis is vertical and whose semi vertical angle is α. If W be the weight and λ the modulus of elasticity of the string, prove that it will be in equilibrium when in the form of a circle whose radius is

$$a\left(1 + \frac{W}{2\pi\lambda}\cot\alpha\right).$$

10

(d) Solve $x^2\dfrac{d^2 y}{dx^2} + 3x\dfrac{dy}{dx} + y = (1 - x)^{-2}$.   10