IFoS - 2013
MATHEMATICS

Time allowed: Three Hours **Maximum Marks:200**

Question Paper Specific Instructions

Please read each of the following instructions carefully before attempting questions.

There are EIGHT questions in all, out of which FIVE are to be attempted.

Question No. 1 and 5 are compulsory. Out of the remaining Six Questions, Three are to be attempted selecting at least ONE question from each of the two Sections A and B.

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PAPER-I

SECTION - A

Q.1 (a) Find the dimension and a basis of the solution space W of the system **(8)**

$x+2y+2z-s+3t=0,\ \ x+2y+3z+s+t=0,\ \ 3x+6y+8z+s+5t=0$

(b) Find the characteristic equation of the matrix $A=\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}$ and hence find the matrix represented by $A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I.$ **(8)**

(c) Evaluate the integral $\int_0^\infty\int_0^x xe^{-x^2/y}\,dy\,dx$ by changing the order of integration. **(8)**

(d) Find the surface generated by the straight line which intersects the lines $y=z=a$ and $x+3z=a=y+z$ and is parallel to the plane $x+y=0.$ **(8)**

(e) Find C of the mean value theorem, if $f(x) = x(x-1)(x-2), a = 0, b = \frac{1}{2}$ and C has usual meaning. **(8)**

Q.2 (a) Let V be the vector space of 2×2 matrices over $\mathbb{R}$ and let $M = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$. Let $F: V \to V$ be the linear map defined by F(A) = MA. Find a basis and the dimension of

(i) the kernel of W of F

(ii) the image U of F **(10)**

(b) Locate the stationary points of the function $x^4 + y^4 - 2x^2 + 4xy - 2y^2$ and determine their nature. **(10)**

(c) Find an orthogonal transformation of co-ordinates which diagonalises the quadratic form.

$q(x, y) = 2x^2 - 4xy + 5y^2.$ **(10)**

(d) Discuss the consistency and the solutions of the equations

$x + ay + az = 1, ax + y + 2az = -4, ax - ay + 4z = 2$ for different values of a. **(10)**

Q.3 (a) Prove that if $a_0, a_1, a_2, \ldots, a_n$ are the real numbers such that

$$\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \ldots\ldots + \frac{a_{n-1}}{2} + a_n = 0$$

then there exists at least one real number x between 0 and 1 such that

$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots\ldots + a_{n-1} x + a_n = 0.$ **(10)**

(b) Reduce the following equation to its canonical form and determine the nature of the conic $4x^2 + 4xy + y^2 - 12x - 6y + 5 = 0.$ **(10)**

(c) Let F be a subfield of complex numbers and T a function from $F^3 \to F^3$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2 + 3x_3, 2x_1 - x_2, -3x_1 + x_2 - x_3)$. What are the conditions on (a, b, c) such that (a, b, c) be in the null space of T ? Find the nullity of T. **(10)**

(d) Find the equations to the tangent planes to the surface $7x^2 - 3y^2 - z^2 + 21 = 0$, which pass through the line $7x - 6y + 9 = 0, z = 3.$ **(10)**

Q.4 (a) Evaluate $\int_0^{\pi/2} \frac{x \sin x \cos x dx}{\sin^4 x + \cos^4 x}.$ **(10)**

(b) Let $H=\begin{bmatrix} 1 & i & 2+i \\ -i & 2 & 1-i \\ 2-i & 1+i & 2 \end{bmatrix}$ be a Hermitian matrix. Find a non-singular matrix ρ such that $p^t H\overline{P}$ is diagonal and also find its signature. **(10)**

(c) Find the magnitude and the equations of the line of shortest distance between the lines

$$\frac{x-8}{3}=\frac{y+9}{-16}=\frac{z-10}{7}$$

$$\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$$ **(10)**

(d) Find all the asymptotes of the curve $x^4-y^4+3x^2y+3xy^2+xy=0.$ **(10)**

SECTION – B

Q.5 (a) Solve $\frac{dy}{dx}+x\sin 2y=x^3\cos^2 y.$ **(8)**

(b) A particle is performing a simple harmonic motion of period T about centre O and it passes through a point P, where OP = b with velocity v in the direction of OP. Find the time which elapses before it returns to P. **(8)**

(c) $\overline{F}$ being a vector, prove that $\text{curl curl }\vec{F}=\text{grad div }\vec{F}-\nabla^2\vec{F}$

where $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}.$ **(8)**

(d) A triangular lamina ABC of density ρ floats in a liquid of density σ with its plane vertical the angle B being in the surface of the liquid, and the angle A not immersed. Find ρ/σ in terms of the lengths of the sides of the triangle. **(8)**

(e) A heavy uniform rod rests with one end against a smooth vertical wall and with a point in its length resting on a smooth peg. Find the position of equilibrium and discuss the nature of equilibrium. **(8)**

Q.6 (a) Solve the differential equation $\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+\left(4x^2-1\right)y=-3e^{x^2}\sin 2x$ by changing the dependent variable. **(13)**

(b) Evaluate $\int_S \vec{F} \cdot d\vec{s}$, where $\vec{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$ and s is the surface bounding the region $x^2 + y^2 = 4$, z = 0 and z = 3. **(13)**

(c) Two bodies of weights w_1 and w_2 are placed on an inclined plane and are connected by a light string which coincides with a line of greatest slope of the plane; if the coefficient of friction between the bodies and the plane are respectively μ_1 and μ_2, find the inclination of the plane to the horizontal when both bodies are on the point of motion, it being assumed that smoother body is below the other. **(14)**

Q.7 (a) Solve $(D^3 + 1)y = e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}}{2}x\right)$

where $D = \frac{d}{dx}$. **(13)**

(b) A body floating in water has volumes v_1, v_2 and v_3 above the surface, when the densities of the surrounding air are respectively ρ_1, ρ_2, ρ_3. Find the value of:

$$\frac{\rho_2 - \rho_3}{v_1} + \frac{\rho_3 - \rho_1}{v_2} + \frac{\rho_1 - \rho_2}{v_3}.$$ **(13)**

(c) A particle is projected vertically upwards with a velocity u, in a resisting medium which produces a retardation kv^2 when the velocity is v. Find the height when the particle comes to rest above the point of projection. **(14)**

8. (a) Apply the method of variation of parameters to solve $\frac{d^2y}{dx^2} - y = 2(1+e^x)^{-1}$. **(13)**

(b) Verify the Divergence theorem for the vector function $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - xz)\vec{j} + (z^2 - xy)\vec{k}$. tTaken over the rectangular parallelopiped $0 \le x \le a, 0 \le y \le b, 0 \le z \le c$. **(14)**

(c) A particle is projected with a velocity v along a smooth horizontal plane in a medium whose resistance per unit mass is double the cube of the velocity. Find the distance it will describe in time t. **(13)**

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PAPER-II

SECTION - A

Q.1 (a) Evaluate $\lim_{x\to 0}\left(\frac{e^{ax}-e^{bx}+\tan x}{x}\right)$ **(10)**

(b) Prove that if every element of a group (G, 0) be its own inverse, then it is an abelian group. **(10)**

(c) Construct an analytic function $f(z)=u(x,y)+\text{i}v(x,y)$, where $v(x,y)=6xy-5x+3$. Express the result as a function of z. **(10)**

(d) Find the optimal assignment cost from the following cost matrix. **(10)**

	A	B	C	D
I	4	5	4	3
II	3	2	2	6
III	4	5	3	5
IV	2	4	2	6

Q.2 (a) Show that any finite integral domain is a field. **(13)**

(b) Every field is an integral domain – Prove it. **(13)**

(c) Solve the following Salesman problem: **(14)**

	A	B	C	D
A	∞	12	10	15
B	16	∞	11	13
C	17	18	∞	20
D	13	11	18	∞

Q.3 (a) Show that the function $f(x) = x^2$ is uniformly continuous in (0, 1) but not in $\mathbb{R}$. **(13)**

(b) Prove that: **(14)**

(i) the intersection of two ideals is an ideal

(ii) a field has no proper ideals.

(c) Evaluate $\oint_C \frac{e^{2z}}{(z+1)^4} dz$ where c is the circle |z| = 3. **(13)**

Q.4 (a) Find the area of the region between the x-axis and $y = (x-1)^3$ from $x = 0$ to $x = 2$. **(13)**

(b) Find Laurent series about the indicated singularity. Name the singularity and give the region of convergence.

$\frac{z - \sin z}{z^3}; z = 0.$ **(13)**

(c) $x_1 = 4, x_2 = 1, x_3 = 3$ is a feasible solution of the system of equations **(14)**

$2x_1 - 3x_2 + x_3 = 8$

$x_1 + 2x_2 + 3x_3 = 15$

Reduce the feasible solution to two different basic feasible solutions.

SECTION - B

Q.5 (a) Use Newton – Raphson method and derive the iteration scheme $x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$ to calculate an approximate value of the square root of a number N. Show that the formula $\sqrt{N} \approx \frac{A+B}{4} + \frac{N}{A+B}$ where AB = N, can easily be obtained if the above scheme is applied two times. Assume A = 1 as an initial guess value and use the formula twice to calculate the value of $\sqrt{2}$ [For 2nd iteration, one may take A = result of the 1st iteration]. **(14)**

(b) Eliminate the arbitrary function f from the given equation $f\left(x^2+y^2+z^2, x+y+z\right)=0$ **(12)**

(c) Derive the Hamiltonian and equation of motion for a simple pendulum. **(14)**

Q.6 (a) Solve the PDE: $xu_x + yu_y + zu_z = xyz$ **(12)**

(b) Convert $(0{\cdot}231)_5$, $(104{\cdot}231)_5$ and $(247)_7$ to base 10. **(12)**

(c) Rewrite the hyperbolic equation $x^2u_{xx} - y^2u_{yy} = 0(x>0, y>0)$ in canonical form. **(16)**

Q.7 (a) Find the values of a and b in the 2-D velocity field $\vec{v} = \left(3y^2 - ax^2\right)i + bxy\, j$ so that the flow becomes incompressible and irrotational. Find the stream function of the flow. **(14)**

(b) Write an algorithm to find the inverse of a given non-singular diagonally dominant square matrix using Gauss – Jordan method. **(13)**

(c) Find the solution of the equation $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 1$ that passes through the circle $x^2 + y^2 = 1, u = 1$. **(13)**

Q.8 (a) Solve the following heat equation, using the method of separation of variables:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, 0 < x < 1, t > 0$$

subject to the conditions

u = 0 at x = 0 and x = 1, for t > 0

u = 4x (1–x), at t = 0 for $0 \leq x \leq 1$. **(16)**

(b) Use the Classical Fourth - order Runge - Kutta method with h = 0.2 to calculate a solution at x = 0.4 for the initial value problem $\frac{du}{dx} = 4 - x^2 + u, u(0) = 0$ on the interval [0, 0.4]. **(12)**

(c) Draw a flow chart for testing whether a given real number is a prime or not. **(12)**